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REAL ANALYSIS

Syllabus :-

03-25 UNIT-1 : Sets, subsets, power set, cartesian product of sets & their countability & their cardinality. Types of sets i.e. finite, infinite, countable, uncountable, similarity of sets or equivalence of sets.

26-33 UNIT-2 :- Real number system, intervals, nbd, limit point, isolated point, adherent point, condensation point, supremum, infimum etc.

34-95 UNIT-3 :- Sequence and series of real numbers

34-82 (A) Definitions

i) sequence
ii) range
iii) subsequence
iv) monotonic sequences
v) limit and convergence of sequences
vi) Cauchy sequence

83-95 (B) Series :-

Definitions, Necessary and sufficient conditions for convergence, types of convergence (convergence, absolute convergence, conditional convergence).

95-125 (C) Test of convergence

126-204 UNIT-4 :- Limit, continuity and differentiability of functions.

207-233

UNIT-5 :- Sequence and series of functions and their uniform convergence (207)

UNIT-6 :- Application of derivatives, monotonicity of functions, Taylor series, some important theorems like Rolle's theorem, Lagrange's mean value theorem (LMVT), Cauchy mean value theorem (CMVT) and expressions.

UNIT-7 :- Riemann Integration (234–253)

UNIT-8 :- Improper integrals (254-273)

UNIT-9 :- Function of several variables (274-285)

UNIT-10 :- Appendix :- (286–

Function of bounded variation, absolute continuity, Cantor set, closed set, Open set, compact set, metric spaces etc.

Keywords For Solving Objective Type Questions

Sequence

Some Useful Facts which are remember :-

1). A sequence $\langle a_n \rangle$ is bdd $\Leftrightarrow \exists$ a $M \in \mathbb{R}^+$ s.t. $|a_n| \leq M \ \forall \ n \in \mathbb{N}$.

2). Every convergent sequence has a unique limit.

3). Every convergent sequence is bdd but not conversely.

4). If $\langle a_n \rangle \to l \Rightarrow \langle |a_n| \rangle \to |l|$ but not conversely.

5). If sequence $\langle a_n \rangle$ diverges to $\infty \Rightarrow \langle a_n \rangle$ is bdd below but not bdd above. Converse is not true.

6). If sequence $\langle a_n \rangle$ diverges to $-\infty \Rightarrow \langle a_n \rangle$ is bdd above but not bdd below. Converse is not true

7). Every monotonically increasing sequence which is bdd above converges to its l.u.b.

8). Every monotonically decreasing sequence which is bdd below converges to its g.l.b.

9). Every monotonically ↑ sequence which is not bdd above diverges to $+\infty$.

10). Every monotonically ↓ sequence which is not bdd below diverges to $-\infty$.

11). Every monotonic sequence either converges or diverges but never oscillates.

12) A sequence $\langle a_n \rangle \longrightarrow 0$ (null sequence) $\Leftrightarrow |a_n| \longrightarrow 0$

13) A sequence $\langle a_n \rangle \longrightarrow l \Leftrightarrow \langle a_n - l \rangle \longrightarrow 0$

14) If $|a_n| \leq |b_n| \; \forall \; n$ and $\langle b_n \rangle \longrightarrow 0 \Rightarrow \langle a_n \rangle \longrightarrow 0$

15) If $\langle a_n \rangle \longrightarrow 0$ and $\langle b_n \rangle \longrightarrow 0 \Rightarrow \langle a_n + b_n \rangle \longrightarrow 0$

16) If $\langle a_n \rangle \longrightarrow 0$ and $\langle b_n \rangle$ is bdd $\Rightarrow \langle a_n \cdot b_n \rangle \longrightarrow 0$

17) If $\langle a_n \rangle$ is oscillately finitely and $\langle b_n \rangle \longrightarrow 0 \Rightarrow \langle a_n b_n \rangle \longrightarrow 0$

18) If $\langle a_n \rangle \longrightarrow 0 \Rightarrow \langle c a_n \rangle \longrightarrow 0$ where c is constant.

19) **Algebra of Limits:**

If $\langle a_n \rangle \longrightarrow a$, $\langle b_n \rangle \longrightarrow b$ then

(i) $\langle k a_n \rangle \longrightarrow ka$

(ii) $\langle |a_n| \rangle \longrightarrow |a|$

(iii) $\langle a_n + b_n \rangle \longrightarrow a+b$

(iv) $\langle a_n - b_n \rangle \longrightarrow a-b$

(v) $\langle a_n b_n \rangle \longrightarrow a \cdot b$

(vi) $\left\langle \frac{1}{a_n} \right\rangle \longrightarrow \frac{1}{a}$; $a \neq 0$

(vii) $\left\langle \frac{a_n}{b_n} \right\rangle \longrightarrow \frac{a}{b}$; $b \neq 0$

20) If $\lim a_n = a \neq 0$, then $\exists$ a $k \in \mathbb{R}^+$ and $m \in \mathbb{N}$ s.t $|a_n| > k \; \forall \; n \geq m$

21) If sequence $\langle a_n \rangle \longrightarrow a$ and $a_n \geq 0 \; \forall \; n$ then $a \geq 0$

22) If sequence $\langle a_n \rangle \longrightarrow a$ and $\langle b_n \rangle \longrightarrow b$ and $a_n \leq b_n \; \forall \; n$ then $a \leq b$.

9mb.

23) If $\langle a_n \rangle$ is a sequence s.t. $a_n \neq 0 \ \forall n$ and $\frac{a_{n+1}}{a_n} = l$ where $|l| < 1$ then $a_n \to 0$

24) Sandwitch Theorem :- If $\langle a_n \rangle$, $\langle b_n \rangle$ & $\langle c_n \rangle$ are 3 sequences s.t. $a_n \leq b_n \leq c_n \ \forall n \geq m$ and $a_n \to l$ and $c_n \to l$ then $b_n \to l$.

25) Cauchy I^st Theorem :- If $a_n \to l$ then $\frac{a_1 + a_2 + \cdots + a_n}{n} \to l$

26) Corollary :- If $a_n > 0 \ \forall n$ and $a_n \to l > 0$ then $(a_1 a_2 \cdots a_n)^{1/n} \to l$.

27) Cauchy end theorem :- If $\langle a_n \rangle$ is a sequence s.t. $a_n > 0 \ \forall n$

$$\lim (a_n)^{1/n} = \lim \left(\frac{a_{n+1}}{a_n} \right)$$

provided the limit on R.H.L exists finitely or infinitely.

28) Caesaro Theorem :- If $\lim a_n = a$, $\lim b_n = b$ then $\lim \left(\frac{a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1}{n} \right) = ab$.

29) If $\langle a_n \rangle$ is convergent and $\langle b_n \rangle$ is divergent then $\langle a_n + b_n \rangle$ is also divergent.

30) If $\langle a_n \rangle$ diverges to ∞ and $b_n > a_n \ \forall n$ then $\langle b_n \rangle$ diverges to ∞.

31) If $\langle a_n \rangle$ diverges to $-\infty$ and $b_n \leq a_n \ \forall n$ then $\langle b_n \rangle$ diverges to $-\infty$.

32). If l is a l.p. of the range $\langle a_n \rangle$ then l is a l.p. of sequence $\langle a_n \rangle$. but not conversely.

33). If a $\langle a_n \rangle \longrightarrow l$ then l is unique l.p. of $\langle a_n \rangle$.

34). B & W Theorem :- Every bdd sequence has at least one l.p.

35). The set of l.p. of a bdd sequence is bdd.

36). Every bdd sequence has the greatest and the least l.p.

37). The set of l.p. of a sequence is a closed set.

38). The set of l.p. of a bdd sequence is a compact set.

39). $\langle a_n \rangle \longrightarrow l \iff \limsup a_n = \liminf a_n = l$

40). If a sequence $\langle a_n \rangle$ s.t. $\overline{\lim}\, a_n = \underline{\lim}\, a_n = \infty$ then $\langle a_n \rangle$ diverges to $+\infty$ and conversely.

41). If a sequence $\langle a_n \rangle$ s.t. $\overline{\lim}\, a_n = \underline{\lim}\, a_n = -\infty$ then $\langle a_n \rangle$ diverges to $-\infty$ and conversely.

42). For a sequence $\langle a_n \rangle$, $\underline{\lim}\, a_n \leq \overline{\lim}\, a_n$

43). If $\langle a_n \rangle$ and $\langle b_n \rangle$ are bdd sequences s.t. $a_n \leq b_n \ \forall n \in \mathbb{N}$ then

(i) $\overline{\lim}\, a_n \leq \overline{\lim}\, b_n$

(ii) $\underline{\lim}\, a_n \leq \underline{\lim}\, b_n$

44). If $\langle a_n \rangle$ and $\langle b_n \rangle$ are bdd sequences then

(i) $\overline{\lim}\,(a_n + b_n) \leq \overline{\lim}\, a_n + \overline{\lim}\, b_n$

(ii) $\underline{\lim}\,(a_n + b_n) \geq \underline{\lim}\, a_n + \underline{\lim}\, b_n$

45). If $\langle a_n \rangle$ is bdd sequence then

(i) $\overline{\lim}\,(-a_n) = -\underline{\lim}\, a_n$

(ii) $\underline{\lim}\,(-a_n) = -\overline{\lim}\, a_n$

(iii) $\underline{\lim}\,(\lambda a_n) = \lambda\, \underline{\lim}\,(a_n)$; $\lambda > 0$

46). If a sequence $\langle a_n \rangle \longrightarrow l$ then every subsequence of $\langle a_n \rangle$ also converges to l <u>but not conversely</u>.

47). If subsequence $\langle a_{2n-1} \rangle$ and $\langle a_{2n} \rangle$ of a sequence $\langle a_n \rangle$ converges to the same limit l then the sequence $\langle a_n \rangle \longrightarrow l$.

48). (i) If a sequence $\langle a_n \rangle$ diverges to ∞ then every subsequences of $\langle a_n \rangle$ also diverges to ∞ <u>but not conversely</u>.

(ii) If a sequence $\langle a_n \rangle$ diverges to $-\infty$ then every subsequences of $\langle a_n \rangle$ also diverges to $-\infty$ <u>but not conversely</u>.

49). Every sequence contains a monotonic subsequence.

50). Every bdd sequence in $\mathbb{R}$ contains a convergent subsequence.

51). If $l \in \mathbb{R}$ is a subsequential limit of $\langle a_n \rangle$
$\Leftrightarrow$ if each nbd of l contains infinitely many terms of $\langle a_n \rangle$
i.e. l is l.p. of $\langle a_n \rangle$.

52). Every convergent sequence is bounded.

53). A sequence is convergent $\Leftrightarrow$ Cauchy sequence

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54) If $\langle a_n \rangle$ is bounded then :-

(i) sequence $\langle a_n \rangle$ has limit pt(s).

(ii) sequence has convergent subsequence.

(iii) Every subsequence is bounded.

(iv) if uniqueness of limit point is confirmed then $\langle a_n \rangle$ is convergent.

We should not say :-

(i) $\langle a_n \rangle$ is convergent

(ii) it has only one or finite no. of limit points unless additionally informed.

(iii) the sequence lies between lim sup & lim inf only

Infact $a_n \notin (\liminf a_n, \limsup a_n)$ for finite values of n only.

We should imply

$\sup S = \limsup a_n$

$\inf S = \liminf a_n$

S is the set of limit points of $\langle a_n \rangle$.

55) If $\langle a_n \rangle$ is unbounded then :-

(i) $\langle a_n \rangle$ may have convergent subsequence.

(ii) $\langle a_n \rangle$ may or may not have limit points.

(iii) $\langle a_n \rangle$ has limit point(s) iff it has cgt subsequence

(iv) $\langle a_n \rangle$ may or may not diverge $\nearrow a_n = n$ $\{1, 2, 3, \dots\}$

(v) $\langle a_n \rangle$ may have finite lim sup. or lim inf but either only.

(vi) $\langle a_n \rangle$ diverges iff $\limsup a_n = \liminf a_n = +\infty$ or $-\infty$

56) If $\langle a_n \rangle$ is monotonic then :- (Monotonic Sequence)

(i) $\langle a_n \rangle$ is either convergent or divergent

(ii) if $\langle a_n \rangle$ is increasing and not convergent then $\langle a_n \rangle$ diverges to $+\infty$

(iii) if $\langle a_n \rangle$ is decreasing and not convergent then $\langle a_n \rangle$ diverges to $-\infty$.

(iv) if $\langle a_n \rangle$ is bdd then convergent.

(v) $\lim a_n = l \Rightarrow \sup a_n = l$ (if a_n is $\uparrow$)

or $\inf a_n = l$ (if a_n is $\downarrow$)

(vi) all the subsequences are monotonic.

(vii) if it has bdd subsequence then $\langle a_n \rangle$ is convergent.

(viii) it has either no limit point or unique l.p.

(ix) every sequence has a monotonic subsequence.

57) Convergent Sequence

If $\langle a_n \rangle$ is convergent sequence then :-

(i) $\langle a_n \rangle$ has unique limit point & that is the limit of the sequence

(ii) all the subsequences have unique limit point and that is same as that of $\langle a_n \rangle$.

(iii) all the subsequences are convergent to the same limit as that of $\langle a_n \rangle$.

(iii) $\langle a_n \rangle$ is bdd along with all the subsequences.

(iv) satisfies algebra of sequences.

(v) if $\langle a_n \rangle$ is cgt & $\langle b_n \rangle$ is not cgt. then $\langle a_n b_n \rangle$ may or may not be convergent.

(vi) if $\langle a_n \rangle$ is convergent & $\langle b_n \rangle$ is not convergent then $\langle a_n + b_n \rangle$ and $\langle a_n - b_n \rangle$ are not convergent.

(vii) if $\langle a_n \rangle$ is convergent then $\lim a_n = a \Rightarrow \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n)$ provided f is continuous at $x = a$.

58). Cauchy sequence :- If $\langle a_n \rangle$ is Cauchy sequence then all the results which holds for convergent sequence are true here also.

V. V. Important Notes which are given below:- :-

1). Every limit point of a sequence $\langle a_n \rangle$ contains infinite members of the sequence in its every nbd but outside the nbd they may be finite or infinite.

2). If l is l.p. of the sequence $\langle a_n \rangle$ s.t. every nbd of l leaves out only finite member of the sequence
(i) the sequence $\langle a_n \rangle$ converges and l becomes limit.
(ii) sequence $\langle a_n \rangle$ cannot have any other l.p.

3). If a sequence has unique l.p. we should not imply sequence is convergent unless we prove it is bdd and sequence is bdd as well as sequence is convergent and the l.p. becomes limit. $\{0_n\}$

4). Monotonic sequences have either no limit point or unique limit. e.g. $1 \cdot n$, $2 \cdot \frac{1}{n}$.

i.e In monotonic sequence if $\exists$ a convergent subsequence then sequence is convergent.

5). If a sequence has convergent subsequence then it has l.p. and conversely.

6). Every subsequences of a convergent sequence is also convergent and converges to the same limit.

7). If sequence is unbounded even then it can have bdd subsequence. Moreover unbounded sequence can have bdd subsequence

8). The sup of a sequence need not be l.u.b. or limit of the sequence and conversely.
Same case in inf.

9). If sequence is monotonic ↑ then $\sup a_n = \lim a_n$.

10). If sequence is monotonic ↓ then $\inf a_n = \lim a_n$.

Sequence	Boundedness	Monotonicity	Limit Point	Nature	Bounds	$\overline{\lim} a_n$	$\underline{\lim} a_n$
$1/n$	bdd sequence	decreasing	$\{0\}$	cgt $\to 0$	$0 < a_n \leq 1 \ \forall n$	0	0
n and n^2	bdd below	increasing	$\times$	dgt $\to \infty$	$\infty > a_n \geq 1 \ \forall n$	∞	∞
$-n$ and $-n^2$	bdd above	decreasing	$\times$	dgt $\to -\infty$	$-\infty < a_n \leq -1 \ \forall n$	$-\infty$	$-\infty$
$(-1)^n$	bdd sequence	not monotonic	$\{-1, 1\}$	finite oscillatory	$-1 \leq a_n \leq 1 \ \forall n$	1	-1
$(-1)^n . n = \begin{cases} n & \text{even} \\ -n & \text{odd} \end{cases}$	neither bdd below nor above	not monotonic	$\times$	infinite oscillatory	$-\infty < a_n < \infty \ \forall n$	$-\infty$	∞
$\frac{(-1)^n}{n} = \begin{cases} 1/n & \text{even} \\ -1/n & \text{odd} \end{cases}$	bdd sequence	not monotonic	$\{0\}$	cgt $\to 0$	$-1 \leq a_n \leq 1/2 \ \forall n$	0	0
$1 + \frac{(-1)^n}{n} = \begin{cases} 1+1/n & \text{even} \\ 1-1/n & \text{odd} \end{cases}$	bdd sequence	not monotonic	$\{1\}$	cgt $\to 1$	$0 \leq a_n \leq 3/2 \ \forall n$	1	1
$\begin{cases} 0 & \text{when } n \text{ is prime or } 1 \\ n & \text{else} \end{cases}$	bdd below	not monotonic	$\{\}$	dgt	$0 \leq a_n < \infty \ \forall n$	∞	∞
$\begin{cases} 0 & \text{when } n \text{ is prime or } 1 \\ p & \text{when } n \text{ is composite} \end{cases}$ (p is the smallest prime divisor)	bdd below	not monotonic	$\{0\} \cup$ set of prime	dgt	$0 \leq a_n < \infty \ \forall n$	∞	∞
c	bdd	monotonic	$\{c\}$	cgt $\to c$	$a_n = c \ \forall n$	c	c
$\frac{n+1}{n}$ or $1+\frac{1}{n}$	bdd	decreasing	$\{1\}$	cgt $\to 1$	$1 \leq a_n \leq 2 \ \forall n$	1	1
$\begin{cases} n & \text{if } n \text{ is even} \\ \frac{1}{n} & \text{if } n \text{ is odd} \end{cases}$	bdd below	not monotonic	$\{0\}$	dgt	$0 < a_n < \infty \ \forall n$	0	∞

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Series

1). $\sum a_n$ is convergent then

(i) $\sum \frac{a_n}{1+a_n}$ is converget.

(ii) $\sum \frac{a_n}{1-a_n}$, is convergent.

2). $\sum a_n$ is convergent $\Rightarrow \sum a_n^2$ is convergent.

3). $\sum a_n$ is divergent $\Rightarrow \sum a_n^2$ may be diverges or converges.

Definition of Limit :- We say $\lim_{x \to a} f(x)$ exist if any two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ converges to 'a' and 'b' then $\langle f(a_n) \rangle$ and $\langle f(b_n) \rangle$ converges to same limit.

→ This method only use for failure

Left Hand Limit (LHL) : We say the function $f(x)$ has L.H.L at $x=a$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $|f(x_1) - f(x_2)| < \varepsilon$ $\forall\ x_1, x_2 \in (a-\delta, a)$

then we write

$\lim_{x \to a^-} f(x)$ exists or $\lim_{h \to 0} f(a-h)$ exists.

R.H.L :- We say the function $f(x)$ has R.H.L at $x=a$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $|f(x_1) - f(x_2)| < \varepsilon$ $\forall\ x_1, x_2 \in (a, a+\delta)$

then we write

$\lim_{x \to a^+} f(x)$ exists or $\lim_{h \to 0} f(a+h)$ exists.

Note :- (i) $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$ iff $\lim_{x \to a} f(x)$ exists.

(ii) Monotonically increasing $\Leftrightarrow$ non-decreasing
" decreasing $\Leftrightarrow$ non-increasing

(iii) Strictly increasing $\Leftrightarrow$ increasing
Strictly decreasing $\Leftrightarrow$ decreasing.

V.Imp
Note:- If $f(x)$ is monotonically in $[a,b]$ & that every point of interval, L.H.L. and R.H.L both the limit exist.

V.Imp.
Note:- If $f(x)$ is increasing function in (a,b) and $c \in (a,b)$ then

$$\lim_{x \to c^+} f(x) = \inf_{x \in (c, c+\delta)} f(x) \quad \text{for some } \delta.$$

$$\lim_{x \to c^-} f(x) = \sup_{x \in (c-\delta, c)} f(x)$$

Imp
Note:- If $f(x)$ is decreasing function in $[a,b]$ and $c \in (a,b)$ then

$$\lim_{x \to c^+} f(x) = \sup f(x)$$

$$\lim_{x \to c^-} f(x) = \inf f(x)$$

Note:- One-one — i) By Graph
ii). Even function, not one-one.

Test For Series

1). Test $\sum_{n=0}^{\infty} x^n$

(i) converges if $-1<x<1$ i.e. $|x|<1$.

(ii) diverges if $x>1$.

(iii) Oscillates finitely for $x=-1$.

(iv) Oscillates infinitely if $x<-1$.

2). **D' Alembert Test** :-

$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = l$ if $l>1$ convergent
$l<1$ divergent
$l=1$ fails.

3). **Cauchy Root Test** :- If Σu_n +ve terms

(a) $\lim (u_n)^{1/n} = l$ then (i) convergent if $l<1$
(ii) divergent if $l>1$
(iii) fails if $l=1$

(b) $\lim (u_n)^{1/n} = \infty \Rightarrow$ divergent

4). **Logarithmic Test** :- If Σu_n +ve terms

$\lim \left[n \log \left(\frac{u_n}{u_{n+1}}\right)\right] = l$ then (i) convergent if $l>1$
(ii) divergent if $l<1$
(iii) fails if $l=1$.

5). **Gauss Test** :- If Σu_n +ve

s.t. $\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + \frac{\alpha_n}{n^{1+\delta}}$ where $\delta>0$
$\langle \alpha_n \rangle$ is bdd sequence

then

(i) convergent if $\lambda>1$ (ii) divergent if $\lambda \le 1$.

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Results :-

1). $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$; $p>0$, (i) convergent if $p>1$
(ii) divergent if $0<p\leq 1$.

2). $\sum_{n=3}^{\infty} \frac{1}{n \log n (\log\log n)^p}$; $p>0$ (i) convergent if $p>1$
(ii) divergent if $0<p\leq 1$

3). $\sum_{n=2}^{\infty} \frac{1}{(\log n)^p}$ → divergent $\forall$ p.

4). $\sum_{n=2}^{\infty} \frac{1}{(n\log n)^p}$ (i) convergent if $p>1$
(ii) divergent if $p\leq 1$

5). $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\log n}$ → conditionally convergent.

6). $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{n(\log n)^p}$ → conditionally convergent.

Some Useful Results On Uniform convergence :-

1) **Abel's Test** :- If (i) $\sum f_n(x)$ is uniformly convergent on $[a,b]$

(ii) $\langle g_n(x) \rangle \downarrow \ \forall\ x \in [a,b]$

(iii) $\exists$ a $k \in \mathbb{R}^+$ s.t. $|g_n(x)| < k \ \forall\ x \in [a,b]$ and then $\sum f_n(x) g_n(x)$ is uniformly convergent on $[a,b]$.

2) **Continuity** :- (i) If a sequence of continuous functions $\langle f_n(x) \rangle$ is uniformly convergent to $f(x)$ on $[a,b]$ then f is continuous on $[a,b]$

$$p \to q \ , \ \sim q \to \sim p$$

i.e f is not continuous on $[a,b]$ then $\langle f_n(x) \rangle$ is not uniformly convergent on $[a,b]$.

(ii) If a series $\sum f_n(x)$ of continuous functions is uniformly convergent to $f(x)$ on $[a,b]$ then $f(x)$ is continuous on $[a,b]$

i.e $\sim q \longrightarrow \sim p$.

e.g :- $\langle f_n(x) \rangle = \langle x^n \rangle \ ; \ x \in [0,1]$

$f_1(x), f_2(x), \cdots, f_n(x), \cdots$ all are continuous in $[0,1]$.

Now,

$$\lim_{n \to \infty} f_n(x) = f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in [0,1) \end{cases}$$

$f(x)$ is discontinuous at $x = 1$.

Hence $\langle f_n(x) \rangle$ is not uniformly convergent in $[0,1]$.

E.g.: Show that the series $x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \cdots\cdots$ is not uniformly convergent on $[0,1]$.

Solution:- Terms of the series are continuous function

Now, $S_n(x) = (1+x^4) - \frac{1}{(1+x^4)^{n-1}}$

$$\lim_{n\to\infty} S_n(x) = S(x) = \begin{cases} 0 & ; x=0 \\ 1+x^4 & ; 0<x\le 1. \end{cases}$$

Hence $S(x)$ is discontinuous at $x=0 \in [0,1]$, then this series is not uniformly convergent in $[0,1]$.

3). **Integration** :-

i). If a sequence $\langle f_n(x)\rangle$ converges uniformly to $f(x)$ on $[a,b]$ and each function f_n is integrable on $[a,b]$ then $f(x)$ is integrable on $[a,b]$ and $\langle \int_a^b f_n(x)\,dx \rangle$ converges uniformly to $\int_a^b f(x)\,dx$.

OR.

If sequence $\langle f_n(x)\rangle$ uniformly converges to $f(x)$ then

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

$$p \to q \Rightarrow \sim q \to \sim p$$

ii). If a series $\sum f_n(x)$ converges uniformly to $f(x)$ on $[a,b]$ and each function $f_n(x)$ is integrable on $[a,b]$ then, $f(x)$ is integrable on $[a,b]$ and $\sum \int_a^b f(x)\,dx$ converges uniformly to $\int_a^b f(x)\,dx$.

OR.

If a series $\sum f_n(x)$ is uniformly converges to $f(x)$ then

$$\sum_{n=1}^{\infty} \int_a^b f(x)\,dx = \int_a^b f(x)\,dx$$ i.e. the series is term by term integration

e.g.:- Show that $\langle f_n(x) \rangle = \langle nxe^{-nx^2} \rangle$; $n \in \mathbb{N}$ is not uniformly convergent on $[0,1]$.

Solution:- $\lim_{n\to\infty} f_n(x) = f(x) = 0 \quad \forall\, x \in [0,1]$

Now, $\int_0^1 f(x)\,dx = 0$

Now, $\int_0^1 f_n(x)\,dx = -\frac{1}{2}\int_0^1 -2nx\, e^{-nx^2}\,dx = -\frac{1}{2}\left[e^{-nx^2}\right]_0^1$

$= -\frac{1}{2}\left[e^{-n} - 1\right]$

$= \frac{1-e^{-n}}{2}$

then $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{2} \neq \int_0^1 f(x)\,dx$

Hence $\langle f_n(x) \rangle$ is not uniformly convergent in $[0,1]$.

e.g.:- Examine for term by term integration the series the sum of whose first n terms is

$n^2x\,(1-x)^n \quad ; \quad 0 \le x \le 1$

Solution:- $S_n(x) = n^2x\,(1-x)^n$

$\lim_{n\to\infty} S_n(x) = S(x) = \lim_{n\to\infty} \frac{n^2x}{(1-x)^{-n}} \quad -\left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{2nx}{-(1-x)^{-n}\, g(1-x)} \quad \left(\frac{\infty}{\infty}\right)$

$= \lim_{n\to\infty} \frac{2x}{(1-x)^{-n}\,[g(1-x)]^2} = 0$

i.e. $S(x) = 0 \quad \forall\, x \in [0,1]$

Now, $\int_0^1 S(x)\,dx = 0$

$$\int_0^1 S_n(x)\,dx = \int_0^1 n^2 x(1-x)^n\,dx$$

$$= \int_0^1 n^2(1-x)(1-1+x)^n\,dx \quad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= n^2 \int_0^1 (x^n - x^{n+1})\,dx = n^2\left[\frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \frac{n^2}{(n+1)(n+2)}$$

$$\lim_{n\to\infty} \int_0^1 S_n(x)\,dx = \lim_{n\to\infty} \frac{n^2}{(n+1)(n+2)} = 1$$

i.e $$\lim_{n\to\infty} \int_0^1 S_n(x)\,dx \neq \int_0^1 S(x)\,dx$$

Hence term by term integration is not justified on $[0,1]$ and also given series is not U.C. on $[0,1]$.

4.) **Differentiation**:- (A) If a sequence of function $\langle f_n(x)\rangle$ is s.t.

(i) each $f_n(x)$ is differentiable on $[a,b]$

(ii) each $f_n'(x)$ is continuous on $[a,b]$

(iii) $\langle f_n(x)\rangle$ converges to $f(x)$ on $[a,b]$

(iv) $\langle f_n'(x)\rangle$ converges uniformly to g on $[a,b]$.

then f is differentiable and $f'(x) = g(x)\ \forall\ x \in [a,b]$.

(B) If a series of functions $\sum f_n(x)$ is s.t.

(i) each $f_n(x)$ is differentiable on $[a,b]$

(ii) each $f_n'(x)$ is continuous on $[a,b]$

(iii) $\sum f_n(x)$ converges to $f(x)$ on $[a,b]$

(iv) $\sum f_n'(x)$ converges uniformly to g on $[a,b]$

then f is differentiable on $[a,b]$ and $f'(x) = g(x)\ \forall\ x \in [a,b]$.

e.g. :- Show that the sequence $\langle f_n(x) \rangle = \frac{nx}{1+n^2x^2}$; $0 \le x \le 1$ can not be differentiable term by term at $x=0$

Solution :- $\lim_{n\to\infty} f_n(x) = f(x) = 0 \quad \forall\ x \in [0,1]$.

$$f'(x) = 0 \Rightarrow f'(0) = 0$$

Now, $$f_n'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{\frac{nh}{1+n^2h^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{n}{1+n^2h^2} = n$$

$f_n'(0) \longrightarrow \infty$ as $n \to \infty$

i.e. $f'(0) \neq \lim_{n\to\infty} f_n'(0)$

Hence $\langle f_n(x) \rangle$ cannot be differentiated term by term at $x=0$.

S.K. RATHORE

Results of Uniform Continuous

1). If f is continuous on (a,b) and $\lim_{x\to a^+} f(x)$ and $\lim_{x\to b^-} f(x)$ exists finitely say l and l' respectively then the function

$$\hat{f}(x) = \begin{cases} f(x) & ; x \in (a,b) \\ l & ; x = a \\ l' & ; x = b \end{cases}$$

Now $\hat{f}(x)$ is defined, is continuous on $[a,b]$ then $\hat{f}(x)$ is also continuous and extension on f.

2). If a function which is continuous on (a,b) and hence continuous extension on (a,b) is uniform continuous.

3). If $f(x)$ is uniform continuous on I and $\langle x_n \rangle$ is a Cauchy sequence then $\langle f(x_n) \rangle$ is also Cauchy sequence.

4). If f is defined on I s.t. the slope of any chord joining any two points of interval is bold then f is uniformly continuous.

i.e $\left| \frac{f(x_2) - f(x_1)}{x_2 - x_1} \right| \leq K$

U.C. Imp.
U.C $\Rightarrow$ Lipschitz function (may or may not be)
Lipschitz function $\Rightarrow$ U.C

such function is Lipschitz function and bounds is called Lipschitz constant.

5). Hence every Lipschitz function is uniform continuous but converse is not true.

Eg. (i): $f(x) = \sqrt{x}$; $x \in (0,\infty)$ is not Lipschitz function but uniform continuous.

Whereas $f(x) = \sqrt{x}$; $x \in (a,\infty)$ is Lip as well as U.C.

e.g (ii) $y = f(x) = x^{1/n}$ is not Lipschitz function but uniform continuous always $\forall n$.

e.g (iii) $f'(x)$ is unbdd $\not\Rightarrow$ U.C.

e.g $f(x) = \sin x^2$; $x \in (0, \infty)$ $\longrightarrow$ not U.C.

6). If f is continuous on $\mathbb{R}$ then it is U.C. on any finite length of interval.

$\longrightarrow$ M_n-test

$\longrightarrow$ $f(x) = \begin{cases} a & x = t \\ b & x \in (\ ) \end{cases}$ is discontinuous

not U.C

not U.C. point

(i) $x = 1/n$

$\lim_{n \to \infty} x = 0$ $\longrightarrow$ point of not U.C.

Note:- If each term U.C. sequence is differentiable and $\langle f_n(x) \rangle$ converges to $f(x)$ uniformly then if $\langle f_n'(x) \rangle$ converges to $g(x)$ uniformly then $g(x) = f'(x)$.

i.e. Whenever a sequence of differentiable function converges to $f(x)$ uniformly then sequence of derivative is uniformly convergent to $f'(x)$.

e.g:- $\langle f_n(x) \rangle = \dfrac{\log(1 + n^3 x^2)}{n^2}$

$\lim_{n \to \infty} f_n(x) = f(x) = 0$

Now, $f_n'(x) = \frac{1}{n^2} \cdot \frac{n^2 \cdot 2n}{1+n^3x^2}$

$\Rightarrow f_n'(x) = \frac{2n^n}{1+n^3x^2}$ is U.C

then $\langle \int f_n'(x)\,dx \rangle = \langle f_n(x) \rangle$ is U.C.

Local maxima:-

(i) $f(x) \leq f(c)$ $\forall$ $x \in (c-\delta, c+\delta)$

(ii) $f'(c)=0$, $f''(c) < 0$

(iii) $f'(c)=0$, +ve, 0, −ve

General $f^{n-1}(c)=0$, $f^n(c) \neq 0$

$n = 2, 3, 4 \ldots$, $n \neq 1$

If n is even :- $f^n(c) < 0$

then maxima.

Inflexion :-

(i) $f'(x)$ does not change its sign in the nbd of c.

(ii) $f''(c) = 0$, $f'''(c) \neq 0$

(iii) The point where the function changes its convexity or concavity is called point of inflexion.

→ (i) Convex in $I \Leftrightarrow f'(x)$ is ↑

(ii) Concave in $I \Leftrightarrow f'(x)$ is ↓

Keywords for solving objective type questions

Countability of Sets

Following sets are countable :-

1). The set of all finite subsets of $\mathbb{N}$; the set of natural numbers.

2). The set of intervals with rational end points.

3). The set of circles in the complex plane having rational radii and centres with rational co-ordinates.

4). If $f: A \xrightarrow{\text{onto}} B$ and if A is countable then the set B is also countable.

5). $A = \{a \in \mathbb{R} : 'a' \text{ is a recurring decimal no.}\}$

6). $B = \{b \in \mathbb{R} : b = \sqrt{p/q} \text{ for some distinct primes } p \& q.\}$

7). The set of range of all the sequences is a countable set of real sequences.

8). The domain of a 1-1 function onto a countable set.

9). The range of an onto function with countable domain.

10). $Q(e^{1/2}, e^{1/3}, e^{1/4}, \dots, e^{1/n}) = \{a_0 + a_1 2^{1/2} + a_2 2^{1/3} \dots ; a_i \in Q\}$

11). $Q(\sqrt{2}i) = \{a + b\sqrt{2}i : a, b \in Q\}$

12). The set of points of removable discontinuities of a function is countable.

13). The set of points of ordinary (first kind) discontinuities of a function is countable.

S.K. RATHORE

Following sets are uncountable :-

1). The set of all polynomials.

2). The set of all subsets of a countably infinite set.

3). The set A-B where A is uncountable but B is countable.

4). $\mathbb{R}(i) = \{a+bi : a, b \in \mathbb{R}\}$

→ Counting ⇒ distinguish but distinguish ⇏ counting

Eg: $\mathbb{R}$

Amit

Set :- A well defined collection of objects is called a set.

By well defined we mean there is no confusion regarding inclusion or exclusion of any object.

$A = \{x : x \text{ is an object and } x \text{ satisfies some rule } T\}$

e.g:- $A = \{x \in \mathbb{N} : 1 < x < 10\}$

$2 \in A$

$2.5 \notin A$

$1 \notin A$

Note :- For any set A, $A \notin A$

A is always subset of A i.e $A \subseteq A$ but $A \notin A$

i.e no set belongs to itself.

Russel Paradox :- Collection of all sets is not a set but not universal set.

i.e. we say there is no set of all sets.

→ Let $S = \{X : X \text{ is a set and } X \notin X\}$. Let S be a set

$\Rightarrow X$ is a set

$\Rightarrow S \notin S$

$\Rightarrow S \in S$

$\therefore S$ is not a set

Hence there is no set of all the sets.

Subset :- Let A and B are sets such that $x \in A \Rightarrow x \in B$ then we say $A \subseteq B$.

Note :- (i) If $x \in A \Rightarrow x \in B$ and $\exists\, y \in B$ and $y \notin A$ then $A \subset B$.

(i.e) If $A \not\subseteq B$ then $\exists$ an element x in A s.t $x \notin B$

e.g. $x \in \mathbb{N} \Rightarrow x \in \mathbb{Z} \Rightarrow x \in \mathbb{Q} \Rightarrow x \in \mathbb{R} \Rightarrow x \in \mathbb{C}$

(iii) ϕ is the subset of every set.

i.e. empty set or void set or null set is the subset of every set.

i.e. for any A, $\phi \subseteq A$.

Cardinality of a set:- The number of elements in a finite set A is defined as the cardinality of A. Whereas for an infinite set we coin new symbols to denote their cardinality and these symbols are transfinite numbers.

i.e. $|A|$ = cardinality of A

= No. of elements of A if A is finite

Card $(\mathbb{N}) = |\mathbb{N}| = \aleph_0$ (aleph nought)

The smallest transfinite number.

→ $\mathbb{N}$ = The smallest infinite set in the sense of cardinality.

→ Cardinality of ϕ is zero.

Powir set:- Let A be a set then

$$P(A) = \{X : X \subseteq A\}$$

→ If $|A| = n$ then $|P(A)| = 2^n$

$\phi = \{\}$ = empty set or void set or null set

$|\phi| = 0$

$P(\phi) = \{\phi\} \Rightarrow |P(\phi)| = 2^0 = 1$

$P(\{\phi\}) = \{\phi, \{\phi\}\} \Rightarrow |P(\{\phi\})| = 2^1 = 2$

Cartesian product of sets:- Let A and B are two sets then the cartesian product of A and B is denoted by $A \times B$

$$A \times B = \{(a,b) : a \in A,\ b \in B\}$$

Results:-

(i) If $|A| = m$ and $|B| = n$ then $|A \times B| = mn$

(ii) $(a,b) \neq (b,a)$

(iii) $(a,b) = (b,a) \iff a = b$

(iv) $A \times B = B \times A$ iff $A = B$

(v) $|A \cap B| = r \Rightarrow |(A \times B) \cap (B \times A)| = r^2$

(vi) $|A \times B| = |B \times A| = |A| \cdot |B|$

Similarity of sets:- A set A is said to be similar to B if there exists one-to-one (one-one onto) correspondence between them.

i.e. Two sets are said to be similar if there exists bijection between them.

→ The other terms of similar sets are equivalent or equinumerous sets (in the sense of cardinality)

e.g. $A = \{1, 2, 3, 4, 5\}$

$B = \{x \in B : x \text{ is vowel}\}$

Then we say the set A and B are cardinally equal and they are similar sets.

Results:-

(i) If two sets are similar then they have the same cardinality or they are cardinally equal.

(ii) Two finite sets are similar iff they have the same cardinality.

(iii) If A and B are sets and $|A| = m$ & $|B| = n$ then
(a) if $m < n$ then there is no onto map from A to B.
(b) if $m > n$ then there is no 1-1 map from A to B.
(iv) If there is no onto map from A to B then we say $|A| < |B|$.

Problem:- Show that there is no onto map from any set to its power set.

Proof:- Let $f: \mathbb{N} \xrightarrow{\text{onto}} P(\mathbb{N})$

for any $n \in \mathbb{N}$

$f(n) \in P(\mathbb{N})$

$\Rightarrow f(n) \subseteq \mathbb{N}$

Let $A = \{r \in \mathbb{N} : r \notin f(r)\}$

$\Rightarrow A \subseteq \mathbb{N} \Rightarrow A \in P(\mathbb{N})$

As f is onto $\exists\ t \in \mathbb{N}$ s.t. $f(t) = A$

$t \in f(t)$

$\Rightarrow t \notin A = f(t)$

or, $t \notin f(t) \Rightarrow t \in A = f(t)$

Hence there is no onto map from any set to its power set

$\therefore$ $\boxed{\aleph_0 < 2^{\aleph_0} = c}$ → continuum

→ There is no cardinal number between $\aleph_0$ and $2^{\aleph_0}$.

→ Any set cannot be similar to its power set.
Hence there is no onto map from $\mathbb{N}$ to $P(\mathbb{N})$.
Hence there is no bijection mapping from $\mathbb{N}$ to $P(\mathbb{N})$

Countable set:- A set is said to be countable if it is similar to a subset of natural number or natural number itself.

→ When a set is similar to natural number itself is called countably infinite.

Note:- If the elements of the set can be labelled by the natural numbers then the set is countable and written as $A = \{a_1, a_2, a_3, \ldots a_n, a_{n+1}, \ldots\}$

$\mathbb{N} \times \mathbb{N}$ = countable

(1,1), (1,2), (1,3), (1,4), ----

(2,1), (2,2), (2,3), (2,4) ----

(3,1), (3,2), (3,3), (3,4) ----

(4,1), (4,2), (4,3), (4,4) ---

Results:-

(i) Every subset of countable set is countable.

(ii) Countable union of countable sets is countable.

(iii) Finite Cartesian product of countable sets is countable.

(iv) If there is 1-1 map from any set A to a countable set then A is countable.

→ 1-1 onto

$f: \mathbb{N} \xrightarrow{1\text{-}1 \text{ onto}} \mathbb{Z}$

$f(n) = (-1)^n \left[\frac{n}{2}\right]$

1	2	3	4	5
0	1	-1	2	-2

$[x] = n \; ; \; n \leq \mathbb{N}$

→ $\mathbb{N} \times \mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, = countable sets.

→ $\mathbb{Q} = \{p/q : q \neq 0, \text{ g.c.d } (p,q) = 1\}$

$\mathbb{Q} = \{(p,q) : q \neq 0\} \subseteq \mathbb{Z} \times \mathbb{Z}$

→ Countable set is also called Denumerable or Enumerable set.

Note:- The set of all finite subsets of $\mathbb{N}$ is countable set.

The set of all rationals is countable i.e. $\mathbb{Q}$ is countable.

Proof:- Let $E_n = \{\frac{m}{n} : m \in \mathbb{N}\}$

then $\cup E_n = \mathbb{Q}^+$ countable

$E_{-n} = \{\frac{m}{-n} : m \in \mathbb{N}\}$

then $\cup E_{n-} = \mathbb{Q}^-$ countable

$\therefore \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\} = \mathbb{Q}$ ($\{0\}$ → singleton set (countable))

Hence $\mathbb{Q}$ is countable.

Another method -

$f: \mathbb{Q}^+ \longrightarrow \mathbb{Z} \times \mathbb{Z}$

$f(p/q) = (p, q)$

$\therefore f: \mathbb{Q}^+ \xrightarrow{1-1} \mathbb{Z} \times \mathbb{Z}$

$\therefore \mathbb{Q}^+$ is countable

$f: \mathbb{Q}^+ \xrightarrow{1-1} \mathbb{Q}^-$

$\therefore \mathbb{Q}^-$ is countable

$\therefore \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\} = \mathbb{Q}$ ($\{0\}$ → singleton set (countable))

Hence $\mathbb{Q}$ is countable.

→ $Q = \{ p/q \; ; \; q \neq 0, \; q \in \mathbb{Z}$ and g.c.d of p & q is 1 —

Q is countable $\because Q \subset \mathbb{Z} \times \mathbb{Z}$ & $\mathbb{Z} \times \mathbb{Z}$ is countable [$\because \mathbb{Z}$ is countable]

→ Set of all polynomials upto degree n with rational co-efficient is countable. hence set of all algebraic no. is countable.

Proof:- Let $A_n = \{ p(x) : p(x)$ is a polynomial of degree$n \}$

Let us suppose

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \in A_n$$

$$= (a_0, a_1, a_2, \cdots a_n) \in Q \times Q \times \cdots \; (n+1) \text{ times}$$

$\Rightarrow$ A_n is countable because Q is countable set.

Also let $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \; t \; ; \; 0 \leq t \leq n$

where t is number of roots of the polynomial

$$R_n = \{ a_0, a_1, a_2, \cdots a_n, t \} \in Q \times Q \times \cdots (n+1) \text{times} \times I$$

$$D = \{ 0, 1, 2, \cdots n \}$$

Set of all roots of polynomial with rational co-efficient is countable.

$\Rightarrow$ Set of all algebraic number is countable.

→ Set of all circles in the cartesian plane with rational radius and centre as rational co-ordinates.

Proof: ~~[illegible]~~ $(a, b, c) \in Q \times Q \times Q$ —

as Q is countable $\Rightarrow Q \times Q \times Q$ is countable

Hence the proof.

→ Set of all unit circles in the complex plane with centre as rational co-ordinate.

Proof:- Co-ordinates of centre $(a,b) \in \mathbb{Q} \times \mathbb{Q}$

as $\mathbb{Q}$ is countable $\Rightarrow \mathbb{Q} \times \mathbb{Q}$ is countable.

Hence the proof.

→ Set of all maps from $\mathbb{N} \to \mathbb{Z}_2 = \{0,1\}$ is uncountable.

Proof:- Let $B = \{ f / f : \mathbb{N} \to \mathbb{Z}_2 = \{0,1\} \}$

$f : \mathbb{N} \to \mathbb{Z}_2$

Cardinality of $\mathbb{N} = \aleph_0$

Cardinality of $\mathbb{Z}_2 = 2$

Number of maps from $\mathbb{N} \to \mathbb{Z}_2 = 2^{\aleph_0}$ (uncountable)

→ Set of all functions from $\mathbb{Z}_2 \to \mathbb{N}$ is countable

Proof:- Let $A = \{ f / f : \mathbb{Z}_2 \to \mathbb{N} \}$

$f : \mathbb{Z}_2 \to \mathbb{N}$

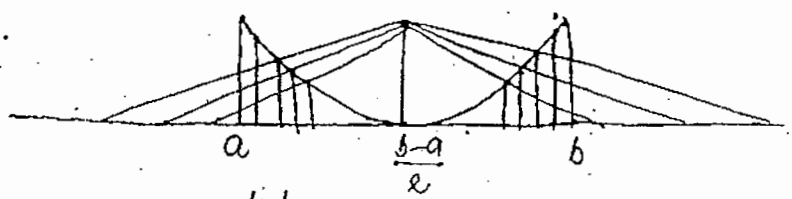
Cardinality of $\mathbb{Z}_2 = \{0,1\} = 2$

Cardinality of $\mathbb{N} = \aleph_0$

Number of maps from $\mathbb{Z}_2 \to \mathbb{N} = \aleph_0^2 = \aleph_0 \times \aleph_0 =$ countable

→ Any interval is similar to whole real numbers

Proof:-



$f : [a,b] \xrightarrow[\text{onto}]{1-1} [c,d]$, $f(a) = c$, $f(b) = d$

Let us take two elements (a,c) and (b,d)

Equation of line

$$y-c = \frac{d-c}{b-a}(x-a)$$

when $x=b$

$\Rightarrow y-c = d-c \Rightarrow y=d$

Thus for every point of x we get y

Hence for each point on semi circle, we get unique point on the real line.

$\therefore [a,b] \sim [c,d]$

Similarly we can prove $(a,b) \sim [c,d]$.

→ If $f: \mathbb{R} \to \mathbb{R}$ and $f(\mathbb{R}) \subset \mathbb{N}$ then f is constant.

Proof:- Let $f: \mathbb{R} \to \mathbb{R}$

for $x \in \mathbb{R}$ $\exists$ $p(t) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$; $a_i \in \mathbb{Q}$

$$p(f(x)) = 0$$

i.e. range set is algebraic

$\therefore$ f is constant.

Questions:-

i). Set of all sequences of 0 and 1 = Uncountable

ii). Set of all disjoint intervals = Countable

iii). $(0,1)$, $[0,1]$, $(0,1]$, $[0,1)$ = Uncountable

iv). Set of all circles in the cartesian plane with rational radius and rational co-ordinates for the centre = countable

v). Set of all polynomials with rational co-efficient = countable

vi). Set of all functions from the set which has two elements $\{0,1\}$ with natural numbers

i.e. $f: \{0,1\} \to \mathbb{N}$ = countable

vii), Set of all functions from the set of natural numbers with the set which has two elements

i.e. $f: \mathbb{N} \to \{0,1\}$ = uncountable.

viii), Set of all algebraic numbers = countable.

ix). Set of all transcedental numbers = uncountable.

x), Let $f: A \to B$ where A and B are infinite sets. Discuss the countability of A or B (whichever is possible) where it is given

(a) f is 1-1

(b) f is onto

Ans:- (a) Let $f: A \xrightarrow{1-1} B$

then $|A| \leq |B|$

∴ When A is uncountable then B is uncountable.
When B is countable then A is countable

(b) Let $f: A \xrightarrow{\text{onto}} B$

then $|B| \leq |A|$

∴ When B is uncountable then A is uncountable
When A is countable then B is countable

Result:- If $f: A \xrightarrow{1-1} B$.

$\Rightarrow$ A is similar to a subset of B

i.e. if there exists 1-1 map from A to B implies A is similar to a subset of B.

Result:- If $f: A \xrightarrow{\text{onto}} B$

$\Rightarrow$ B is similar to a subset of A.

i.e. if there exists onto map from A to B implies B is similar to a subset of A.

Function:-

Definition:- A function from a non-empty set A to another set B is a rule that associates each element of A to some unique element of B.

Types of functions:- Let $f: A \to B$

1) **Injective OR One-one function**:- If f is one-one

i.e $f(a) = f(b)$

$\Rightarrow a = b \quad \forall\ a, b \in A$

2) **Surjective or Onto function**:- If f is onto

i.e. for every $y \in B \ \exists\ x \in A$ s.t. $f(x) = y$.

3) **Bijection**:- If f is one-one and onto both.

Note:- (i) $f(A) = \{f(x) : x \in A\}$ is called range of f.

(ii) $f: [a, b] \to \mathbb{R}$ then $f[a, b]$ is range of f.

Result :- If A and B are finite sets then total number of functions from A to B = $|B|^{|A|}$.

i.e if $f: A \to B$ (n, m) and $|A| = n$ & $|B| = m$ then total number of functions from A to B = m^n

$= |\text{co-domain}|^{|\text{domain}|}$

Result :- If $f: A \to B$ (n, m) and $|A| = n$, $|B| = m$

then

Case (i) :- $n < m$

(i) Total number of onto maps = 0

(ii) Total number of one-one maps = ${}^mP_n = {}^mC_n \cdot \lfloor n$

$= \frac{m!}{(m-n)!} = m(m-1)(m-2)\cdots(m-n+1)$

and $|f(A)| = |A|$ in (ii)

(iii) Total number of bijections = 0.

Case (ii) :- $n > m$

(i) Total number of one-one maps = 0

(ii) Total number of onto maps = $m^n - \{{}^mC_1(m-1)^n - {}^mC_2(m-2)^n + {}^mC_3(m-3)^n - \cdots\}$

$= m^n - \sum_{r=0}^{\infty} (-1)^r \, {}^mC_r \, (m-r)^n$

(iii) Total number of bijections = 0

$f: A \to B$ (5, 2) (n, m)

$2^5 - \{{}^2C_1(2-1)^5\} = 2^5 - {}^2C_1$

$= 32 - 2 = 30$ ↓ Total no. of onto maps

(i) (ii)

* ${}^nC_r = {}^nC_{n-r}$

Case (iii):- $n = m$.

(i) Total number of one-one maps $= n!$

(ii) Total number of onto maps $= n!$

(iii) Total number of bijections $= n!$

Hence between two finite sets there is bijection iff they have the same cardinality.

Lemma:- Every infinite set has a countable subset or countably infinite subset in particular.

Theorem:- Every infinite set is similar to at least one of its proper subset.

Proof:- Let X be an infinite set. By previous lemma $\exists\ A \subseteq X$ such that A is countable (infinite)

$\Rightarrow A = \{a_1, a_2, \cdots a_n, a_{n+1}, \cdots\}$

$\mathbb{N} \to \mathbb{Z} \to \mathbb{Q}$
$\mathbb{N} \subset \mathbb{Z}$
$\mathbb{Z} \subset \mathbb{Q}$

Construct, $A_1 = A - \{a_1\}$

Again construct $Y = (X - A) \cup A_1$

$\Rightarrow a_1 \notin Y$

$\Rightarrow Y$ is proper subset of X

Let us define a map f as

$$f: X \longrightarrow Y$$

$$\text{s.t.} \quad f(x) = \begin{cases} a_{i+1}; & \text{if } x \in A \ \& \ x = a_i \\ x; & x \in X - A \end{cases}$$

It can be easily shown that $f(x)$ is one-one, onto.

Hence f is a bijection from X to Y

$\Rightarrow X$ is similar to Y i.e $\boxed{|X| = |Y|}$

Problem:- What is basic difference between finite set and infinite set ?

Ans:- Every finite set is not similar to at least one of its proper subset whereas every infinite set is similar to at least one of its proper subset.

Problem:- Show that the set of real numbers is uncountable.

OR, Show that the set of all sequences of 0 & 1 is uncountable.

Proof:- Let $A = (0,1) \subset \mathbb{R}$ is countable.

Then we can levelled the element of A as

$$A = (0,1) = \{A_1, A_2, A_3, \cdots, A_n, A_{n+1}, \cdots\}$$

where $A_1 = 0.a_{11}\, a_{12}\, a_{13} \cdots a_{1n} \cdots$

$A_2 = 0.a_{21}\, a_{22}\, a_{23} \cdots a_{2n} \cdots$

$A_3 = 0.a_{31}\, a_{32}\, a_{33} \cdots a_{3n} \cdots$

$\vdots$

$A_n = 0.a_{n1}\, a_{n2}\, a_{n3} \cdots a_{nn} \cdots$

$\vdots$

We take $B = 0.b_1\, b_2\, b_3 \cdots b_n \cdots$

s.t $b_1 \neq a_{11}$

$b_2 \neq a_{22}$

$b_3 \neq a_{33}$

$\vdots$

$b_n \neq a_{nn}$

$\Rightarrow B \neq \bigcup A_i$ for any i

and as B lies between 0 and 1

$\Rightarrow B \notin A = (0,1)$ which is contradiction

$\Rightarrow$ We can not levelling the element of A

$\Rightarrow A = (0,1)$ is uncountable

$\Rightarrow \mathbb{R}$ is uncountable.

Since $(0,1) \sim \mathbb{R}$.

Hence the set of all real numbers is uncountable.

i.e. the set of all sequences of 0 & 1 is uncountable.

$\Rightarrow$ Every interval is uncountable

$\Rightarrow$ Set of all irrational numbers is uncountable

Because $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$

$\mathbb{Q}$ → countable

$\mathbb{Q}^c$ → uncountable

$\Downarrow$

uncountable.

Result:- If A is uncountable, $B \subset A$ s.t. B is countable then $A - B$ is uncountable

i.e. if we remove countable subset from the uncountable set then the resulting set is uncountable.

Problem:- The set of all disjoint intervals is countable.

Proof:- Let A = set of all disjoint intervals

$I, J \in A$

$I \cap J = \phi$

$Q = \{q_1, q_2, q_3, \ldots q_n, q_{n+1}, \ldots\}$

Let $I \in A$

$\exists\ X \subset Q$ such that $X \subset I$

Let r be the smallest natural number s.t. $q_r \in I$

Let $f: A \to Q$

$f(I) = q_r$

$\therefore$ f is 1-1

But we know that if there exists 1-1 map from A to Q implies A is similar to a subset of Q but Q is countable

$\therefore$ A is countable

i.e the set of all disjoint intervals is countable.

Algebraic number:- A real number is said to be algebraic over Q if it can be root of a polynomial equation with rational co-efficient.

→ **Monic polynomial**:- A polynomial is said to be monic polynomial if its leading co-efficient is 1.

→ α is an algebraic number

$P(x) = \sum_{i=1}^{n} a_i x^i \ ; \ a_i \in Q$

such that $P(x) = 0$

$$P_1(x) = \frac{1}{a_n} P_n(x) \quad \text{[monic polynomial]}$$

$$P_1(\alpha) = 0$$

→ If α is an algebraic number then ∃ unique monic polynomial of least degree whose zeros is α.

$$\phi : \underset{\text{algebraic number}}{A} \to P \quad (P \hookrightarrow \text{monic polynomial})$$

$$\phi(\alpha) = P_n(\alpha)$$

ϕ is one-one

Hence set of all algebraic numbers is countable.

$$\boxed{\mathbb{R} = A + T}$$

Where A = set of all algebraic numbers

T = set of all transcedental numbers.

→ For any finite set A to any infinite set B

$$f : \underset{\text{finite}}{A} \xrightarrow[\text{guaranteed}]{1-1} \underset{\text{infinite}}{B}$$

Transcedental number:- A number which is not algebraic is called transcedental number.

e.g. e, π.

Results:-

(i) From countable set to uncountable set there always exist a one-one map but never onto.

(ii) From uncountable set to countable set there always exist a onto but never one-one.

Examples:- (a) $f: Q \longrightarrow Q^c$ (1-1) always exist.

$f: Q^c \longrightarrow Q$ (onto) always exist.

(b) $f: \mathbb{N} \longrightarrow P(\mathbb{N})$ (1-1) always exist

$f: P(\mathbb{N}) \longrightarrow \mathbb{N}$ (onto) always exist.

Result:- If $f: A \xrightarrow{1-1} B$. Then

(i) B is countable $\Rightarrow$ A is countable

(ii) A is uncountable $\Rightarrow$ B is uncountable.

(iii) if B is uncountable $\Rightarrow$ A cannot be determined.

(iv) if A is countable $\Rightarrow$ B cannot be determined.

e.g. $f: \mathbb{N} \xrightarrow{1-1} P(\mathbb{N})$

Result:- If $f: A \xrightarrow{\text{onto}} B$. Then

(i) A is countable $\Rightarrow$ B is countable

(ii) B is uncountable $\Rightarrow$ A is uncountable

(iii) if A is uncountable $\Rightarrow$ B cannot be determined

(iv) if B is countable $\Rightarrow$ A cannot be determined.

e.g. $f: P(\mathbb{N}) \xrightarrow{\text{onto}} \mathbb{N}$.

Problem:- $f: A \to B$ such that range is countable.

(i) If f is onto then A and B are countable

(ii) If f is one-one then A is uncountable.

(iii) If f is one-one then A is countable

(iv) If f is one-one then A may be countable or uncountable.

Problem:- $f: A \to B$, f cannot be onto and A & B are infinite sets then

(i) A is countable ✗

(ii) B is countable ✗

(iii) A is uncountable

(iv) B is uncountable

Problem:- $f: A \to B$, f cannot be one-one and A & B are infinite sets then $|A| > |B|$

(i) A is countable ✗

(ii) B is countable ✗

(iii) A is uncountable ✓

(iv) B is uncountable

Some important facts:-

(i) The intersection of two uncountable sets need not be uncountable.

e.g. $(0,1] \cap [1,2) = \{1\} \to$ Finite = countable

(ii) If there is no one-one map from $\mathbb{N} \to A$

$\Rightarrow$ A is finite.

v.imp (iii) If there is no onto map from $\mathbb{N} \to A$
$\Rightarrow$ A is uncountable

i.e. If $\exists$ no onto map from $\mathbb{N} \to A$ then $\mathbb{N}$ is always similar to a proper subset of A but never similar to itself then A is uncountable.

(iv) $A = \{(a,b) : a,b \in \mathbb{R}\}$ = uncountable $\sim \mathbb{R} \times \mathbb{R}$

(v) $A = \{(a,b) : a,b \in \mathbb{Q}\}$ = countable $\sim \mathbb{Q} \times \mathbb{Q}$

(vi) $A = \{(a,b) : a,b \in \mathbb{Z}\}$ = countable $\sim \mathbb{Z} \times \mathbb{Z}$

v.imp (vii) $[0,1) \cap T$ = uncountable,
where T = the set of all transcedental numbers.

(viii) $f : [0,1) \longrightarrow (0,1]$.
$f(x) = 1 - x$
Then f is one-one and onto.

(ix) $f : [0,1] \longrightarrow (0,1)$
— $A = \{0, 1, 1/2, 1/3, \cdots 1/n, \cdots\}$
Then f is one-one and onto.

(x) If $f : A \xrightarrow[\text{onto}]{1-1} \mathbb{N}$ then A is countable.

(xi) Power sets of any infinite set is uncountable.

(xii) If $f : \mathbb{Z}_m \longrightarrow \mathbb{N}$ (countable)

(xiii) $A = \{f : A \to \mathbb{N}$ and A is finite$\}$
$\Rightarrow$ A is countable.

→ $A = [0,1] - \{0, 1, 1/2, \dots\}$

$A \subset [0,1] \longrightarrow (0,1)$

$$f(x) = \begin{cases} 1/2 & ; x = 0 \\ 1/n+2 & ; x = 1/n \\ n & ; \text{else } x \in A \end{cases}$$

→ $f: (-\pi/2, \pi/2) \longrightarrow \mathbb{R}$

$f(x) = \tan x$

Results:-

Existence of map:-

(i) From any countable set to uncountable set we can not find onto map.

eg $f: \mathbb{N} \longrightarrow \mathbb{P}(\mathbb{N})$

(ii) There does not exist 1-1 map from uncountable set to countable set.

e.g. $f: \mathbb{P}(\mathbb{N}) \longrightarrow \mathbb{N}$

(iii) It is also possible that A and B are two uncountable sets and there is 1-1 map or there is no onto map but not both.

→ $f: (0,1) \longrightarrow (0,1) \cap \mathbb{Q}$

Let $A = (0,1) =$ uncountable

$B = (0,1) \cap \mathbb{Q} =$ countable

$f: A \longrightarrow B$ (no 1-1 map)

$f: B \longrightarrow A$ (no onto map)

→ $x = (p/q)^{1/n} \Rightarrow x^n = p/q \Rightarrow qx^n - p = 0$

→ $A = \{\sqrt{p} : p \text{ is prime}\}$ is countable

→ $\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n$ (transcedental)

→ $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

put $x = i$

$$\log(1+i) = i - \frac{1}{2}(i)^2 + \frac{1}{3}(i)^3 - \frac{1}{4}(1)^4 + \cdots$$

$$\Rightarrow \log(1+i) = i\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \right) + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} + \cdots\right)$$

$$\Rightarrow \log\left(\sqrt{2}\, e^{i\pi/4}\right) = i\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \right) + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots\right)$$

$$\Rightarrow \log\sqrt{2} + i\frac{\pi}{4} = i\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \right) + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} \cdots\right)$$

$$\therefore \boxed{1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}}$$

→ $e + \pi$ is algebraic or transcedental number if it is not proof.

→ $Q^c \cap A$ is countable where A = the set of algebraic numbers.

→ $A = \{\frac{p}{q}\pi : p \in N\}$ is countable.

→ $A = \{a\pi : a \in A\}$ is countable.

→ $A = \{\sqrt[n]{p/q} : g.c.d(p,q) = 1\}$ is countable

→ $A = (0, 1)$, $B = (1/2, 3/2)$

$A - B = (0, 1/2)$ ←

$A \cap B = (1/2, 1)$ ←

$B - A = (1, 3/2)$ ← Uncountable

$A \cup B = (0, 3/2)$ ←

UNIT-2

Point Set Topology

Neighbourhood (nbd):- Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$ then we say S is a nbd of 'a' if $\exists$ an open interval I s.t. $a \in I \subseteq S$.

i.e S is a nbd of $a \in \mathbb{R}$ if $\exists\ \varepsilon > 0$ s.t.

$$(a-\varepsilon, a+\varepsilon) \subseteq S.$$

Properties:-

(i) Since a set which is nbd of some element contains an interval as a subset and intervals are uncountable and hence nbd is uncountable. i.e countable set can not be nbd of any real number. $a \in I \subseteq S$ since I is uncountable then S is also uncountable.

(ii) Since every nbd of any element has to contain an interval, so all those sets which can not contain interval as a subset can not be nbd of any real number.

i.e $I \not\subseteq S$ then S can not be nbd of any $a \in \mathbb{R}$.

e.g. set of irrationals ($\mathbb{Q}^c$), algebraic and transcendental numbers cannot be nbd of any real number.

(iii) As every set is subset of itself, hence for any $\varepsilon > 0$

$I = (a-\varepsilon, a+\varepsilon)$ i.e. $a \in I \subseteq I$ is a nbd of 'a'.

Hence I is a nbd of 'a'.

Or, for $\delta > 0$, $(a-\delta, a+\delta)$ is a δ-nbd of 'a'.

Vimp

Thus every interval is a nbd of all of its ...

Deleted δ-nbd of 'a' :- $(a-\delta, a+\delta) - \{a\}$.

Remarks :-

(i) For any $\varepsilon > 0$ $(a-\varepsilon, a+\varepsilon)$ is a nbd of 'a' but not conversely.

(ii) A set which can not contain interval as a subset cannot be nbd of any real number.

e.g. $\mathbb{Q}$, $\mathbb{N}$, $\mathbb{Z}$.

(iii) ϕ is not a nbd of any real number.

(iv) $\mathbb{Q}^c$ and transcedental numbers are uncountable but not nbd of any real number because it does not contain an interval.

i.e. uncountable set may or may not be nbd of any real number.

Adherent Point :- Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$ then we say 'a' adheres S or 'a' is an adherent point of S or S adheres 'a' if for any $\delta > 0$

$$(a-\delta, a+\delta) \cap S \neq \phi$$

i.e. every nbd of 'a' contains at least one elt of S.

e.g. $A = \{1, 2, 3, 4\}$

$$(4-\delta, 4+\delta) \cap A \neq \phi$$

Properties :-

(i) Every member of the set is an adherent point but not conversely

i.e. adherent point need not be a member of the set

e.g. $(0, 1) = 0, 1$ and all the members of the interval are adherent points.

Limit point or Cluster point or Accumulation point of a set :- Let $a \in \mathbb{R}$ and $S \subseteq \mathbb{R}$ then we say 'a' is a limit point of S if every deleted nbd of 'a' contains at least one element of S.

i.e for any $\varepsilon > 0$.

$$S \cap \{(a-\varepsilon, a+\varepsilon) - \{a\}\} \neq \phi$$

Since for ε we have infinite choices so we can redefine the limit point as follow:

A real number 'a' is said to be limit point of S if every nbd of 'a' contains infinite elements of S.

Remark :-

(i) Finite set has no limit point.

(ii) Infinite set need not have limit point.

e.g. $\mathbb{N}$, $\mathbb{Z}$

(iii) $a \in \mathbb{R}$ and $Q \subseteq \mathbb{R}$

$(a-\varepsilon, a+\varepsilon) \cap Q$ infinite set

i.e Every real number is a limit point of Q

Similarly every real number is a limit point of Q^c.

(iv) Every limit point is an adherent point but not conversely. e.g $\mathbb{N}$, $\mathbb{Z}$, finite set.

(v) Limit point may not be belong to that set

e.g. $A = \{1, 1/2, 1/3, \ldots\}$

limit point = 0 but $0 \notin A$.

Derived set of S:- Let $S \subseteq \mathbb{R}$, then the derived set of S is the collection of all limit points of S and denoted by S'.

i.e. $S' = \{a \in \mathbb{R} : 'a' \text{ is a l.p. of } S\}$

$\mathbb{N}' = \phi$

$\mathbb{Z}' = \phi$

$\mathbb{Q}' = \mathbb{R}$

$(\mathbb{Q}^c)' = \mathbb{R}$

$\mathbb{R}' = \mathbb{R}$

$\mathbb{C}' = \mathbb{C}$

$\mathbb{A}' = \mathbb{R}$ Where A is the set of all algebraic numbers.

$\mathbb{T}' = \mathbb{R}$ Where T is the set of all transcedental numbers.

$(0,1)' = [0,1]$

$[0,1)' = [0,1]$

$(0,1]' = [0,1]$

$[0,1]' = [0,1]$

$\because A = \mathbb{Q} \cap (0,1)$

$A' = [0,1]$

$\because S = \{1, 2, 3\}$

$S' = \phi$

$\because S = \{m + n\sqrt{7}$ [redacted]

$S = \{m + nq ; m, n \in \mathbb{Q}\}$ $q \in \mathbb{Q}^c$ (q any irrational)

then $S' = \mathbb{R}$

$\rightarrow S = \{\frac{1}{m} + \frac{1}{n} ; m, n \in \mathbb{N}\}$

$S' = \{\frac{1}{n} ; n \in \mathbb{N}\} \cup \{0\}$

$\rightarrow S = \{t + \frac{1}{n} ; n \in \mathbb{N}, t \in A\}$

$S' = A \cup A'$

$\rightarrow S = \{m + \frac{1}{n} ; m, n \in \mathbb{N}\}$

$S' = \mathbb{N}$

$\rightarrow S = \{x + \frac{1}{n} ; n \in \mathbb{N}, x \in \mathbb{Q}\}$

$S' = \mathbb{Q} \cup \mathbb{Q}' = \mathbb{Q} \cup \mathbb{R} = \mathbb{R}$

$\rightarrow S = \{n + \frac{1}{n} ; n \in \mathbb{N}\}$

$S' = \phi$

$\rightarrow S = \{x + 1/n : n \in \mathbb{N}, x \in (0,1)\}$

$S' = [0,1]$

$\rightarrow S = \{\cos n\pi : n \in \mathbb{N}\}$

$S' = [-1, 1]$

$\rightarrow S = \{m + n\sqrt{2} : m, n \in$ [redacted]

$S' = \mathbb{R}$.

$\rightarrow S = \{x + \frac{1}{n} ; n \in \mathbb{N}, x \in A = \{1, 2, 3\}\}$

$S' = \{1, 2, 3\}$

→ $S = \{1 + \frac{1}{n} ; n \in \mathbb{N}\}$ → $A = \{\frac{m}{n} : m, n \in \mathbb{N}\}$

$S' = \{1\}$. $A' = [0, \infty)$

Radius of convergence :-

(i) $\frac{1}{R} = \lim_{n \to \infty} \sup \left| \frac{a_{n+1}}{a_n} \right|$

(ii) $\frac{1}{R} = \lim_{n \to \infty} \sup |a_n|^{1/n}$ ✓ prefer it.

Bounded set :- Let $S \subseteq \mathbb{R}$ then S is said to be bounded if $\exists\ m, M \in \mathbb{R}$ s.t $m \leq x \leq M\ \forall\ x \in S$

m is called a lower bound

M is called an upper bound

then $S \subseteq [m, M]$

i.e <u>every bounded set is a subset of an interval of finite length.</u>

Any real number $> M$, is also an upper bound.

Any real number $< m$, is also a lower bound.

Supremum of a set :- The least upper bound of a set if exists, is called the supremum of that set.

If $\sup S = M$ or $x \leq M\ \forall\ x \in S$

then for any $\varepsilon > 0$, $\exists\ x \in S$ s.t. $x > M - \varepsilon$.

Infimum of a set :- The greatest lower bound (g.l.b) of a set if exists, is called the infimum of that set

If $\inf S = m$ or $x \geq m\ \forall\ x \in S$

then for any $\varepsilon > 0$, $\exists\ x \in S$ s.t. $x < m + \varepsilon$.

Note:- The supremum and the infimum of a set need not to be a member of that set.

e.g. (i) $S = \left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$

$= \left\{ 2, 1/2, 1+\frac{1}{3}, 1-\frac{1}{4}, 1+\frac{1}{5}, \cdots \right\}$

$\sup S = 2$

$\inf S = 1/2$

(ii) $S = \left\{ \sin \frac{1}{n} : n \in \mathbb{N} \right\}$ $\longrightarrow S = \left\{ \cos \frac{1}{n} : n \in \mathbb{N} \right\}$

$= \left\{ \sin 1, \sin \frac{1}{2}, \sin \frac{1}{3}, \cdots \right\}$ $\sup S = \cos 0 = 1$

$\inf S = \cos 1$

$\sup S = \sin 1$

$\inf S = 0$

(iii) $S = \{ \sin n\pi : n \in \mathbb{N} \}$

$= \{ 0, 0, 0, \cdots \}$

$\sup S = 0$

$\inf S = 0$

(iv) $S = \{ x \in \mathbb{Q} : x^2 < 5 \}$

$= (-\sqrt{5}, \sqrt{5}) \cap \mathbb{Q}$

$\sup S = \sqrt{5}$

$\inf S = -\sqrt{5}$

Completeness property of $\mathbb{R}$:- Every non-empty subset of real number has supremum if bounded above and infimum if bounded below in $\mathbb{R}$.

→ $(\mathbb{Q}, \leq)$ = incompleteness property of $\mathbb{Q}$ but completeness property of $\mathbb{R}$.

Archemedian property:- For any $\varepsilon > 0$, $\exists\ n \in \mathbb{N}$

s.t $\frac{1}{n} < \varepsilon$; $n > 1, in$

In particular $n = \lceil \frac{1}{\varepsilon} \rceil$

e.g. $|a-b| < \varepsilon \Rightarrow a = b$

* $(\mathbb{N}, \leq)$, $(\mathbb{N}, |)$ = Partially ordered set

Partially order relation = Reflexive, Antisymmetric & Transitive.

Isolated point:- Let $a \in S$ and $S \subseteq \mathbb{R}$ then we say 'a' is an isolated point of S if $\exists$ a nbd of 'a' which contains no element of S other than 'a'.

i.e. $\exists\ \delta > 0$ s.t. $(a-\delta, a+\delta) \cap T = \phi$

where $T = S - \{a\}$

Remarks:-

(i) All the members of a finite set are isolated points.

(ii) Every members of $\mathbb{N}$ or $\mathbb{Z}$ is isolated points.

(iii) $\underbrace{\text{Adherent points} - \text{Limit points}}_{S \cup S'} = \underbrace{\text{Isolated points}}_{S}$

(iv) $\mathbb{R}$ has no isolated points.

(v) Every member is isolated point of the set S then S is countable.

i.e the collection of all isolated points of any set is a countable set.

e.g:- The set of all disjoint intervals.

Bolzano & Weierstrass theorem for sets:- Every infinite and bounded set has a limit point (sufficient condition)

Proof:- Given S is bounded $\Leftrightarrow \exists\ m, M \in \mathbb{R}$

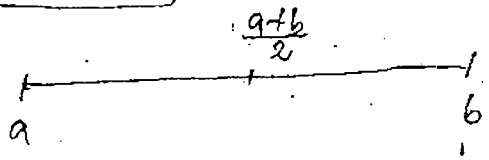
s.t. $m \leq x \leq M\ \forall\ x \in S$

$S \subseteq [m, M]$

Step I:- S is bounded $\Rightarrow S \subseteq [a, b] = I_1 \cup I_2$

Step II:- S is infinite $\Rightarrow$ either $[a, \frac{a+b}{2})$ or $[\frac{a+b}{2}, b]$ contains infinite members of S.

say $[a, \frac{a+b}{2}] = I_1 = [a_1, b_1]$



$I_0 = [a, b]$

$I_1 = [a_1, b_1] \Rightarrow |I_1| = \frac{b-a}{2}$

$I_2 = [a_2, b_2] \Rightarrow |I_2| = \frac{b-a}{2^2}$

$\vdots$

$I_n = [a_n, b_n] \Rightarrow |I_n| = \frac{b-a}{2^n} \to 0$ as $n \to \infty$

$\therefore I_n = \{l\}$ as $n \to \infty$

i.e $\lim_{n\to\infty} I_n = \{l\}$

Remark:- For an infinite set boundedness is the sufficient condition to have a limit point. but not necessary.

e.g:- $\mathbb{Q}^c$ is infinite set and unbounded but it has uncountable limit points. i.e $\mathbb{R}$

UNIT-3 :

Sequences & Series of real numbers

Sequence:- A sequence of real numbers is the collection of images of some map $f: \mathbb{N} \to \mathbb{R}$ when they are arranged in the natural order of natural numbers.

i.e A map $f: \mathbb{N} \to \mathbb{R}$ defines a sequence of real number when it is written like $\{f(1), f(2), \ldots f(n), \ldots\}$ and we adopt a symbol

$f(n) = a_n$; the nth term of the sequence

and write

$$\langle a_n \rangle = \{a_1, a_2, \ldots a_n, a_{n+1}, \ldots\}$$

Examples:-

1). $a_n = (-1)^{n-1} = \{1, -1, 1, -1, \ldots\}$

$l.u.b = sup = 1$
$g.l.b = inf = -1$
} $\langle a_n \rangle$ is bounded

$\langle a_n \rangle$ is not $\uparrow$ and not $\downarrow$

$\Rightarrow \langle a_n \rangle$ is not monotonic

$\Rightarrow \langle a_n \rangle$ is oscillating finitely.

Limit points of $\langle a_n \rangle = 1, -1$

2). $a_n = (-1)^n = \{-1, 1, -1, 1, \ldots\}$ and then Same as (1)

3). $a_n = (-1)^{n+1} = \{1, -1, 1, -1, \ldots\}$ same as (1)

4). $a_n = \frac{1}{n} = \{1, 1/2, 1/3, 1/4, \cdots\}$

l.u.b = sup = 1
g.l.b = inf = 0
} $\langle a_n \rangle$ is bounded

$\langle a_n \rangle$ is strictly decreasing

$\Rightarrow \langle a_n \rangle$ is monotonic

Limit point of $\langle a_n \rangle = 0$

---

5). $a_n = 1 - \frac{(-1)^n}{n} = \{1+1, 1-\frac{1}{2}, 1+\frac{1}{3}, \cdots\}$

l.u.b = sup = 2
g.l.b = inf = $-1/2$
} $\langle a_n \rangle$ is bounded

$\langle a_n \rangle$ is not monotonic.

Limit point of $\langle a_n \rangle = 1$.

---

6). $a_n = n = \{1, 2, 3, \cdots\}$

No l.u.b = No sup
g.l.b = 1
} $\langle a_n \rangle$ is not bounded i.e. unbounded

$\langle a_n \rangle$ is strictly increasing

$\Rightarrow \langle a_n \rangle$ is monotonic.

$\langle a_n \rangle$ has no limit point.

7). $a_n = -n = \{-1, -2, -3, \ldots\}$

l.u.b = sup = −1
No g.l.b
} not bounded i.e unbounded

$\langle a_n \rangle$ is strictly decreasing
$\Rightarrow$ $\langle a_n \rangle$ is monotonic
$\langle a_n \rangle$ has no limit point.

---

8). $a_n = (-1)^n n = \{-1, 2, -3, 4, -5, 6, -7, 8, \ldots\}$

No lub & No glb $\Rightarrow$ not bounded
$\langle a_n \rangle$ is not monotonic
$\Rightarrow$ $\langle a_n \rangle$ oscillates infinitely.
$\langle a_n \rangle$ has no limit point.

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9). $a_n = \begin{cases} 0, & \text{when } n \text{ is prime or } 1 \\ n, & \text{when } n \text{ is composite} \end{cases}$

$\langle a_n \rangle = \{0, 0, 0, 4, 0, 6, 0, 8, 9, \ldots\}$

no l.u.b
g.l.b = 0
} $\langle a_n \rangle$ is not bounded.

$\langle a_n \rangle$ is not monotonic
$\Rightarrow$ $\langle a_n \rangle$ Oscillates
Limit point of $\langle a_n \rangle = 0$
i.e 0 is only l.p. of $\langle a_n \rangle$.

10). $a_n = \begin{cases} 0 & ; \text{when } n \text{ is prime or } 1 \\ p & ; \text{when } n \text{ is composite and } p \text{ is the smallest prime dividing } n. \end{cases}$

$a_{p^n} = p \quad \text{for } n > 1$

$\langle a_n \rangle = \{0, 0, 0, 2, 0, 2, \ldots\}$

Limit points of $\langle a_n \rangle$ = All primes p and 0
i.e infinite limit points.

$glb = \inf = 0$, No sup $\Rightarrow \langle a_n \rangle$ is not bounded
i.e $\langle a_n \rangle$ is bdd below

$\langle a_n \rangle$ is not monotonic

$\Rightarrow \langle a_n \rangle$ oscillates

---

11). $a_{n+2} = a_n + a_{n+1} \quad ; a_1 = 1, a_2 = 1$

$= \{1, 1, 2, 3, 5, 8, 13, 21, \ldots\}$

$glb = \inf = 1$, no lub $\Rightarrow$ is not bounded
i.e bdd below.

$\therefore$ Given sequence is monotonic

$\Rightarrow$ Given sequence oscillates.

$\Rightarrow$ Given sequence has no limit points.

---

12). $a_{n+2} = \frac{1}{2}(a_n + a_{n+1})$, a_1 and a_2 are given.

$a_1 < a_2$

13). $a_n = c$; c is constant

$\langle a_n \rangle = \{c, c, c, \cdots c, \cdots\}$

l.u.b = g.l.b = c $\Rightarrow$ $\langle a_n \rangle$ is bounded

$\Rightarrow$ $\langle a_n \rangle$ is monotonic

Limit point of $\langle a_n \rangle = c$.

---

14). $a_n = \begin{cases} p & ; n = p^r, r > 1 \text{ \& } p \text{ is prime} \\ 0 & ; \text{else} \end{cases}$

$\langle a_n \rangle = \{0, 0, 0, 2, 0, 0, 0, 0, 3, 0, 0, 0 \cdots 0, 0, \cdots 0\}$

g.l.b = 0.
no l.u.b $\Rightarrow$ $\langle a_n \rangle$ is unbounded

$\Rightarrow$ $\langle a_n \rangle$ is not monotonic

Limit points of $\langle a_n \rangle$ all primes : p and 0.

i.e infinite limit points of $\langle a_n \rangle$.

---

15). $a_{n+1} = \sqrt{2 + a_n}$; $a_1 = 0$

$\langle a_n \rangle = \{0, \sqrt{2}, \sqrt{2+\sqrt{2}}, \cdots\}$

inf = g.l.b = 0
sup = l.u.b = 2 $\Rightarrow$ $\langle a_n \rangle$ is bounded

$\Rightarrow$ $\langle a_n \rangle$ is monotonic $\uparrow$ sequence

Limit point of $\langle a_n \rangle = 2$.

i.e $\lim_{n \to \infty} \langle a_n \rangle = 2$.

Range of a sequence:- The range of a map or function which defines the sequence is called the range of the sequence.

Remarks:-

(i) If range is singleton $\Rightarrow$ sequence is constant.

(ii) If range of a sequence is finite then the sequence has at least one member which repeats itself infinitely many times.

i.e. $\exists\ l \in \mathbb{R}$ s.t. $a_n = l$ for infinite values of n.

(iii) Range of a sequence is always a countable set.

Bounded sequence:- A sequence $\langle a_n \rangle$ is said to be bounded if $\forall\ a_n$, $m \le a_n \le M$ $\forall\ n \in \mathbb{N}$

i.e. a sequence is said to be bounded if its range is bounded.

i.e. $m, M \in \mathbb{R}$

s.t. $m \le a_n \le M$ $\forall\ n \in \mathbb{N}$

Bounded above sequence:- A sequence $\langle a_n \rangle$ is said to be bounded above if $\exists\ M \in \mathbb{R}$ s.t. $a_n \le M$ $\forall\ n \in \mathbb{N}$.

Bounded below sequence:- A sequence $\langle a_n \rangle$ is said to be bold below if $\exists\ m \in \mathbb{R}$ s.t. $a_n \ge m$ $\forall\ n \in \mathbb{N}$.

Remark:- Range of a bounded sequence is a bounded set and conversely.

Lemma:- If a range of a sequence is finite
ie Range $\langle a_n \rangle = S$ and S is finite
$\exists\ \alpha \in S$ such that $a_n = \alpha$ for infinite values of $n \in \mathbb{N}$

Note:- But not conversely. eg: $\{0, \frac{1}{2}, 0, \frac{1}{3}, 0, \frac{1}{4}, \dots\}$
$\lim a_n = 0$, but range set is not finite

Monotonic sequence:- Sequence $\langle a_n \rangle$ is said to be monotonic if

(Non-decreasing) Monotonic ↑: $a_n \le a_{n+1}$ $\forall\ n \in \mathbb{N}$

(Non-increasing) Monotonic ↓: $a_n \ge a_{n+1}$ $\forall\ n \in \mathbb{N}$

Increasing = strictly increasing
Decreasing = strictly decreasing.

Note:- Constant sequences are the only sequences which are monotonic increasing as well as monotonic decreasing.
ie constant sequences are monotonic.
* [Strictly means no term repeats.]

Limit point of a sequence:- Let $\langle a_n \rangle$ be a sequence of real numbers and $l \in \mathbb{R}$, we say l is a limit point of $\langle a_n \rangle$ if for any $\varepsilon > 0$
$a_n \in (l-\varepsilon, l+\varepsilon)$ for infinite values of n

e.g: (i) $\langle (-1)^{n-1} \rangle = \{1, -1, 1, -1, \dots\}$

$(1-\varepsilon, 1+\varepsilon)$ & $(-1-\varepsilon, -1+\varepsilon)$

Limit points = $1, -1$ [∵ 1 and -1 repeats infinitely many times].

(ii) $a_n = \frac{1}{n} \to 0$ is l.p.

Range $a_n = S$ (bounded and infinite)

$\exists\ l \in \mathbb{R}$ s.t. l is a limit point of S.

$\rightarrow$ $a_n = \frac{1}{1-n}$ is not a sequence because a_1 is not defined.

Bolzano and Weierstrass theorem for sequence:-

Every bounded sequence has a limit point. (Sufficient condition)

Proof:- Let the sequence $\langle a_n \rangle$ is bounded

$\Rightarrow$ Range $\langle a_n \rangle = S$ is bounded. $S_R = \{a_1, a_2, a_3, \ldots\}$, $1-1=0$

Case I: When S is infinite and bounded.

By B & W for set, S has a limit point say l

$\Rightarrow \langle a_n \rangle$ has a limit point l.

$\Rightarrow$ Hence the proof.

Case II:- When S is finite and bounded.

$\exists\ k \in \mathbb{R}$ such that $a_n = k$ for infinite values of n.

$\Rightarrow \langle a_n \rangle$ has a limit point k.

Hence the proof.

Note:- If range set of a sequence has limit point then these limit points of range set are also the limit points of the sequence whereas the limit point of the sequence need not be a limit point of the range. eg (1)

Imp

Result:- If sequence $\langle a_n \rangle$ is monotonic and bounded then $\langle a_n \rangle$ is convergent.

Convergence of a sequence / Limit of a sequence:-

Let $l \in \mathbb{R}$ and $\langle a_n \rangle$ be a sequence we say the sequence $\langle a_n \rangle$ converges to 'l' if for any $\varepsilon > 0$, $\exists\, m \in \mathbb{N}$ (depending on ε) s.t. $|a_n - l| < \varepsilon$ whenever $n \geq m$ (or $\forall\, n \geq m$) and then 'l' is called the limit of the sequence $\langle a_n \rangle$ and expressed as

$$\boxed{\lim_{n\to\infty} a_n = l} \quad \text{or} \quad \boxed{\lim a_n = l}$$

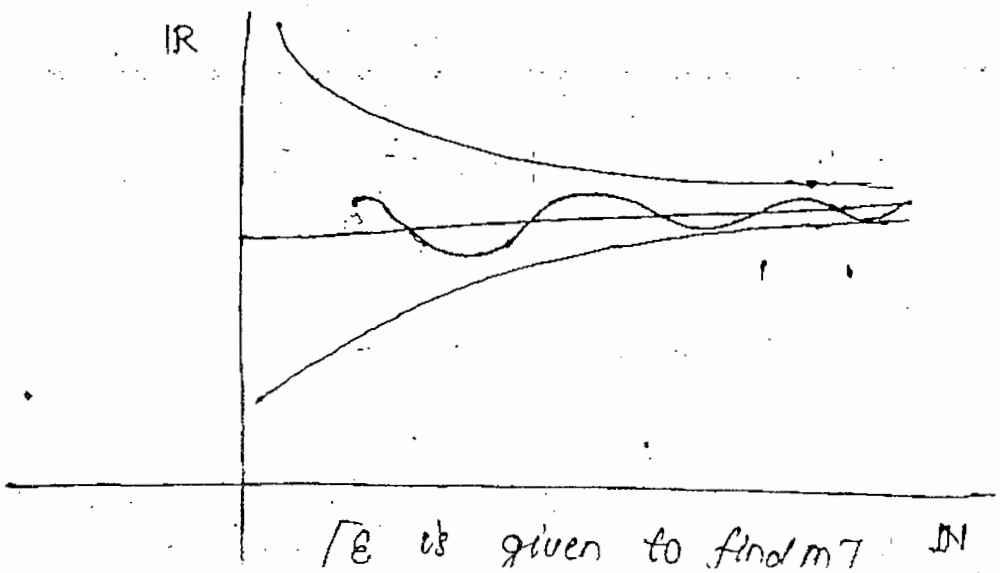
i.e a real number 'l' is said to be the limit of a sequence $\langle a_n \rangle$ iff the distance between a_n and 'l' can be made as small as we please.

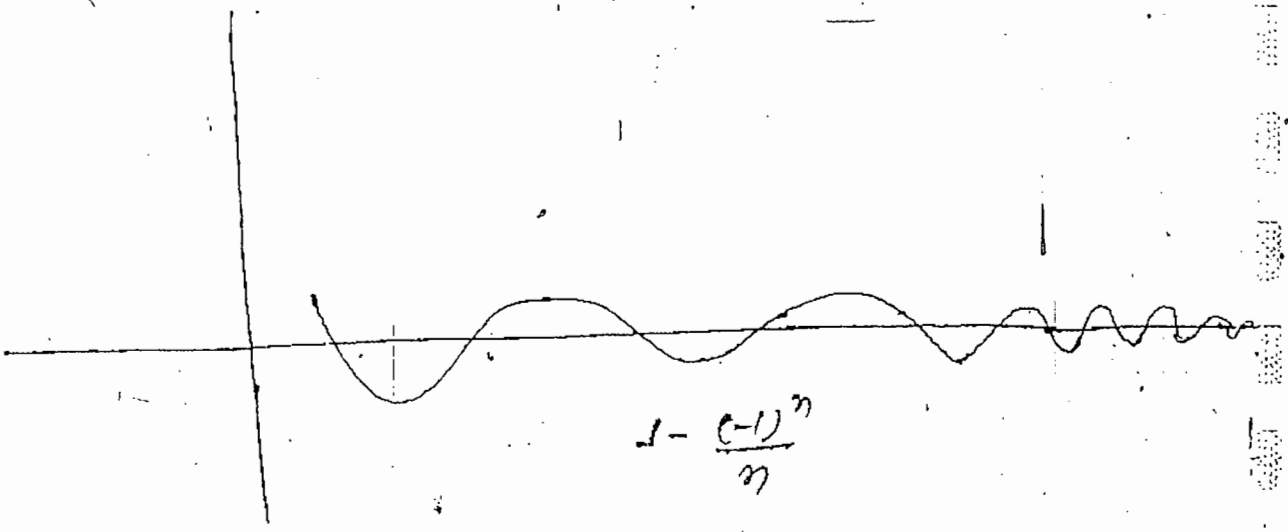
i.e. a real number is said to be the limit of a sequence if every nbd of that real number leaves out only finite members of the sequence.

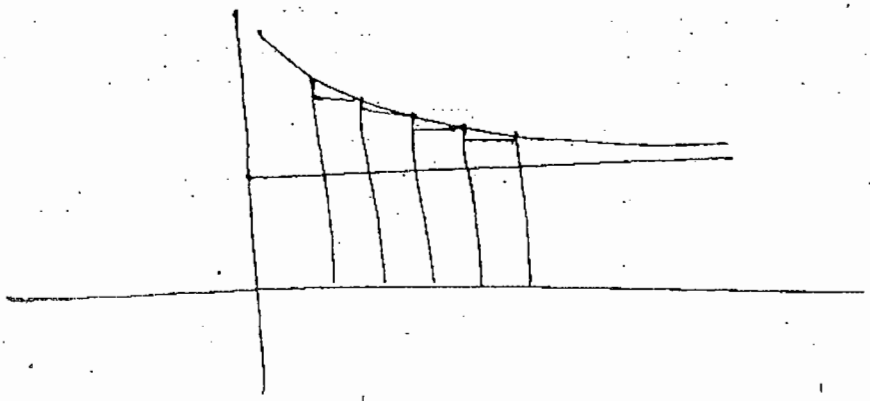
$$-\varepsilon < a_n - l < \varepsilon \;;\; n \geq m$$

$$\Rightarrow\; l - \varepsilon < a_n < l + \varepsilon \;;\; n \geq m$$

$$\Rightarrow\; a_n \in (l-\varepsilon) \;;\; n \geq m \quad (m \text{ is finite})$$







Algorithm:-

Write $|a_n - l| < \varepsilon$

and try to find the inequality of the type

$n > \phi(\varepsilon)$

s.t. $\phi(\varepsilon)$ is defined for all $\varepsilon > 0$ and bounded for all $\varepsilon > 0$ then $m = \lceil \phi(\varepsilon) \rceil$

OR, $|a_n - l| < |\psi(n)| < \varepsilon$

$n > \phi(\varepsilon)$

$m = \lceil \phi(\varepsilon) \rceil$

Example :- $\langle a_n \rangle$ s.t. $a_n = \sqrt{n+1} - \sqrt{n}$

$\lim a_n = 0$

Let $\varepsilon > 0$ be given

$$|\sqrt{n+1} - \sqrt{n-0}| = \frac{1}{\sqrt{n+1}+\sqrt{n}} < \frac{1}{\sqrt{n}+\sqrt{n}} = \frac{1}{2\sqrt{n}} < \varepsilon$$

$$\Rightarrow n > \frac{1}{4\varepsilon^2}$$

$$m = \left\lceil \frac{1}{4\varepsilon^2} \right\rceil \rightarrow \text{Ceiling function}$$

$\Rightarrow |\sqrt{n+1} - \sqrt{n}| < \varepsilon$ whenever $n > m$.

Result :- The limit of a sequence is the limit point of that sequence but not conversely.

Proof :- Let l be the limit of the sequence $\langle a_n \rangle$ then for any $\varepsilon > 0$, $\exists\ m \in \mathbb{N}$

$$\text{s.t. } |a_n - l| < \varepsilon \ ; \ n \geq m$$

or, $a_n \in (l-\varepsilon, l+\varepsilon) \ ; \ n \geq m$

$\Rightarrow$ 'l' is a limit point of the sequence $\langle a_n \rangle$.

Hence proof.

Result :- Whenever a sequence is convergent it converges to a unique limit.

Proof :- Let $\lim a_n = l_1$ and also $\lim a_n = l_2$.

Then for any $\varepsilon > 0$, $\exists\ m_1$

$$\text{s.t. } |a_n - l_1| < \varepsilon/2 \ ; \ n \geq m_1$$

and $\exists\ m_2$ s.t $|a_n - l_2| < \varepsilon/2 \ ; \ n \geq m_2$

Now, $|l_1 - l_2| = |l_1 - a_n + a_n - l_2|$

$$\leq |a_n - l_1| + |a_n - l_2|$$

$< \varepsilon/2 + \varepsilon/2$ whenever $n = m = \max\{m_1, m_2\}$.

$\Rightarrow |l_1 - l_2| < \varepsilon$

$\Rightarrow \boxed{l_1 = l_2}$ Hence proved.

Note:- (i) Every convergent sequence has unique limit point but converse is not true. e.g. $\{0, 1, 0, 2, 0, 3, \ldots\}$ $l = 0$

(ii) Unique limit point does not imply sequence is convergent.

(iii) Every convergent sequence is bounded but not conversely.

Proof: Let $\langle a_n \rangle$ be a sequence which converges to l.

We shall show that $\langle a_n \rangle$ is a bounded sequence.

Let $\varepsilon > 0$ be any given number.

Since $\langle a_n \rangle$ converges to l, therefore there exists a +ve integer m s.t. $|a_n - l| < \varepsilon \ \forall\ n \geq m$

$\Rightarrow l - \varepsilon < a_n < l + \varepsilon \ \forall\ n \geq m$

Let $t = \min\{a_1, a_2, \ldots a_{m-1}, l - \varepsilon\}$

& $T = \max\{a_1, a_2, \ldots, a_{m-1}, l + \varepsilon\}$

$\Rightarrow t < a_n < T \ \forall\ n \in \mathbb{N}$

$\therefore$ Sequence $\langle a_n \rangle$ is bounded.

But converse is not true.

e.g. Consider the sequence $\langle a_n \rangle$ where $a_n = (-1)^{n-1}$

$\langle a_n \rangle = \{1, -1, 1, -1, \ldots\}$

The sequence $\langle a_n \rangle$ is bounded but not convergent.

Result:- (i) A bounded sequence is convergent iff it has unique limit point.

i.e. bdd + unique l.p ⇔ convergent.

(ii) A sequence has a limit point 'l' (say) iff there is a subsequence converging to 'l'.

Note:- For any real number 'l' (say) we can construct a sequence of desired type (rational, irrational, algebraic, transcedental) which converges to that real number 'l'.

$l \in \mathbb{R}$

$a_n \in (l - \frac{1}{n}, l + \frac{1}{n})$

$\Rightarrow |a_n - l| < 1/n$

and for any $\varepsilon > 0$, $\exists\, m \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$, $n > m$

$\Rightarrow |a_n - l| < \varepsilon$, $n > m$

$\Rightarrow \langle a_n \rangle \longrightarrow l$.

$\rightarrow \langle (-1)^n \rangle \Rightarrow$ l.p. = 1 & −1 infinitely many times.

$\rightarrow \left\langle 1 + \frac{(-1)^n}{n} \right\rangle \begin{cases} n > 4 \\ n < 4 \end{cases}$ finite times l.p = 1

$\rightarrow \left\langle 1 - \frac{1}{n} \right\rangle$ no times l.p = 1.

Prob $\longrightarrow$ $a_n, b_n \in (\alpha - \frac{1}{n}, \alpha + \frac{1}{n}) \cap \mathbb{Q}$

$\langle a_n \rangle$ & $\langle b_n \rangle \longrightarrow \alpha$, $a_n, b_n \in \mathbb{Q}$

Case I:- $\alpha \in \mathbb{Q}^c \Rightarrow a_n = \frac{p_n}{q_n}$ then $\langle p_n \rangle \longrightarrow \infty$

$\langle q_n \rangle \longrightarrow$

Case (ii):- $\alpha \in \mathbb{Q} \Rightarrow b_n = \frac{p_n'}{q_n'}$ then $\langle p_n' \rangle \to \infty$
$\langle q_n' \rangle \to \infty$ or $q=0$

If $\alpha \in \mathbb{Q} - \{0\} \Rightarrow b_n = \frac{p_n'}{q_n'}$ then $\langle p_n' \rangle \to \infty$
$\langle q_n' \rangle \to \infty$

Subsequence:- Let f be an strictly increasing function from $\mathbb{N}$ to $\mathbb{N}$
i.e. $f: \mathbb{N} \to \mathbb{N}$
then $\langle f(k) \rangle$ defines an strictly increasing sequence of natural numbers say $\langle n_k \rangle$

$$\langle n_k \rangle = \{ n_1, n_2, \dots n_r, n_{r+1}, \dots \}$$
$$n_1 < n_2 < \dots < n_r < n_{r+1} < \dots$$

e.g (i) $f(k) = 2^k$
$n_k = 2^k$

(ii) $f(k) = 2k = \{2, 4, 6, 8, \dots\}$
$f(k) = 2k-1 = \{1, 3, 5, 7, 9, \dots\}$

Let $\langle a_n \rangle$ be a sequence of real numbers and $\langle n_k \rangle$ be strictly increasing sequence of natural numbers then $\langle a_{n_k} \rangle$ is defined as a subsequence of $\langle a_n \rangle$.

eg:- $\langle a_{2k} \rangle$, $\langle a_{2k-1} \rangle$ are subsequences of $\langle a_n \rangle$.

Result:- Criteria for convergence:- If subsequences of odd and even terms respectively converges to the same limit then the sequence is convergent.

Proof:- We have given

$$\lim_{2r\to\infty} a_{2r} = \lim_{2r-1\to\infty} a_{2r-1} = l \quad \text{---(1)}$$

We have to show that $\lim_{n\to\infty} a_n = l$.

By definition of (1)

$$|a_{2r} - l| < \varepsilon \quad \forall \; 2r > m_1$$

$$|a_{2r-1} - l| < \varepsilon \quad \forall \; 2r-1 > m_2$$

Let $m = \max\{m_1, m_2\}$

$\Rightarrow |a_n - l| < \varepsilon \quad \forall \; n > m$

$\Rightarrow \langle a_n \rangle$ is convergent and converges to l (say).

Hence proved.

Results:- (i) If sequence $\langle a_n \rangle$ is increasing and bounded above then it is convergent and converges to its supremum.

Proof:- $\langle a_n \rangle$ is increasing and bounded above.

Let $\sup \langle a_n \rangle = l$

For any $\varepsilon > 0$, $\exists\, k \in \mathbb{N}$ s.t. $a_k > l - \varepsilon$

$\Rightarrow a_n > l - \varepsilon \quad \forall \; n > k$.

as $\sup \langle a_n \rangle = l$

$\Rightarrow a_n < l \quad \forall \; n \in \mathbb{N}$

$\Rightarrow a_n < l+\varepsilon \quad \forall n \in \mathbb{N}$

$l-\varepsilon < a_n < l+\varepsilon \; ; n>k$

$\Rightarrow |a_n - l| < \varepsilon \; ; n>k$

(ii) If sequence $\langle a_n \rangle$ is decreasing and bounded below then it is convergent and converges to its infimum.

(iii) If sequence $\langle a_n \rangle$ is monotonic and bounded then it is convergent.

(iv) Bounded and unique limit point then it is cgt.

Proof:- If sequence $\langle a_n \rangle$ is bounded then $\exists\, l \in \mathbb{R}$ s.t. l is a l.p. of $\langle a_n \rangle$

For any $\varepsilon > 0$ $a_n \in (l-\varepsilon, l+\varepsilon)$ for infinite values of n and $a \notin (l-\varepsilon, l+\varepsilon)$ only for finite values of n else.

Some special Examples:-

Example (1):- $\langle a_n \rangle$ whenever $a_n = \sqrt{3a_n}$, $a_1 = 1$

$a_2 = \sqrt{3}$

$a_3 = \sqrt{3\sqrt{3}}$

$a_2 < a_3$

$\sqrt{3a_{k+1}} < \sqrt{3a_k}$

$a_k < a_{k+1} \Rightarrow \langle a_n \rangle$ is ↑ sequence

$a_k < 3$

$a_{k+1} = \sqrt{3a_k} < \sqrt{3 \cdot 3} = 3$

$\Rightarrow a_{k+1} < 3$

$\Rightarrow$ sequence $\langle a_n \rangle$ is bdd.

$\Rightarrow \langle a_n \rangle$ is bdd & ↑ sequence

$\Rightarrow \langle a_n \rangle$ is convergent.

To find limit :-

Let $\langle a_n \rangle$ converges to l

$$a_{n+1} = \sqrt{3a_n}$$

$$\lim a_{n+1} = \lim \sqrt{3a_n}$$

$$l = \sqrt{3l}$$

$$\Rightarrow l^2 = 3l$$

$$\Rightarrow l^2 - 3l = 0$$

$$\Rightarrow l(l-3) = 0$$

$$\Rightarrow l = 0, 3$$

$$\Rightarrow \boxed{\lim a_n = 3} \quad (0 \text{ can't be limit point.})$$

Example (2) :- $a_{n+1} = \sqrt{2 + a_n}$

$a_1 = 0$ (given)

$a_2 = \sqrt{2 + a_1} = \sqrt{2}$

$a_3 = \sqrt{2 + \sqrt{2}}$

$$a_1 < a_2 < a_3$$

Let us assume that sequence is increasing for $n \geq k$

$$a_{k-1} < a_k$$

$$\sqrt{2 + a_{k-1}} < \sqrt{2 + a_k}$$

$$a_k < a_{k+1}$$

Hence it is true for $\forall n$

Hence, sequence is increasing sequence

Also let $a_k < 2$

$$a_{k+1} = \sqrt{2 + a_k}$$

$a_{k+1} < \sqrt{2+2}$

$a_{k+1} < 2$.

Hence given sequence is bounded.

Hence given sequence is convergent because it is monotonic and bounded.

To find limit :-

$$a_{n+1} = \sqrt{2+a_n}$$

$$\lim a_n = l$$

$$\Rightarrow \quad l = \sqrt{2+l}$$

$$\Rightarrow \quad l^2 - l - 2 = 0$$

$$\Rightarrow \quad l^2 - 2l + l - 2 = 0.$$

$$\Rightarrow \quad l(l-2) + 1(l-2) = 0$$

$$\Rightarrow \quad (l-2)(l+1) = 0$$

$$\Rightarrow \quad l = -1, 2$$

$$\therefore \quad \boxed{\lim a_n = 2} \qquad [\because a_1 = 0, \text{ & } 0 < a_n < 2]$$

Example :- ③ $a_{n+2} = \frac{1}{2}(a_n + a_{n+1})$ and a_1 & a_2 are given. Show that $\langle a_n \rangle$ is convergent and find limit.

Solution :- **Case (i) :-** If $a_1 = a_2$

$$a_3 = \frac{1}{2}(a_1 + a_2) = \frac{1}{2}(a_1 + a_1) = a_1$$

$$\Rightarrow \langle a_n \rangle = \langle a_1, a_1, - - - \rangle$$

$$\Rightarrow \langle a_n \rangle \text{ is constant sequence}$$

$$\Rightarrow \langle a_n \rangle \text{ is convergent.}$$

Case (ii) :- If $a_1 < a_2$

$$a_3 = \frac{1}{2}(a_1 + a_2)$$

$\Rightarrow a_1 < a_3 < a_2$

$a_4 = \frac{1}{2}(a_2 + a_3)$

$\Rightarrow a_1 < a_3 < a_4 < a_2$

Similarly $a_1 < a_3 < a_5 < a_4 < a_2$

$\Rightarrow a_1 < a_3 < a_5 < \cdots\cdots < a_6 < a_4 < a_2$

$\langle a_{2r-1} \rangle = a_1 < a_3 < a_5 \cdots\cdots$ / bdd by a_2

$\langle a_{2r} \rangle = a_2 > a_4 > a_6 \cdots\cdots$ / bdd by a_1

Both the sequences $\langle a_{2r} \rangle$ and $\langle a_{2r-1} \rangle$ are monotonic and bounded

$\Rightarrow \langle a_{2r} \rangle$ and $\langle a_{2r-1} \rangle$ are convergent.

Now we have to show that limit of both are same and hence given sequence is convergent.

Let $\langle a_{2r-1} \rangle \to l_1$ and $\langle a_{2r} \rangle \to l_2$

As $a_{n+2} = \frac{1}{2}(a_n + a_{n+1})$

Taking limit we get

$l_1 = \frac{1}{2}(l_1 + l_2) \Rightarrow l_1 = l_2 = l$ (say)

or, $l_2 = \frac{1}{2}(l_2 + l_1) \Rightarrow l_1 = l_2 = l$ (say)

$\Rightarrow \lim \langle a_{2r-1} \rangle = \lim \langle a_{2r} \rangle = l$

$\Rightarrow$ Given sequence $\langle a_n \rangle$ is convergent and converges to l (say).

Case (iii):- If $a_1 > a_2$

Same as case (ii)

To find limit :-

$$a_3 = \frac{1}{2}a_1 + \frac{1}{2}a_2$$

$$a_4 = \frac{1}{2}a_2 + \frac{1}{2}a_3$$

$$a_5 = \frac{1}{2}a_3 + \frac{1}{2}a_4$$

$$\vdots$$

$$a_{n-1} = \frac{1}{2}a_{n-2} + \frac{1}{2}a_{n-3}$$

$$a_n = \frac{1}{2}a_{n-1} + \frac{1}{2}a_{n-2}$$

Adding we get

$$a_3 + a_4 + a_5 + \cdots + a_{n-2} + a_{n-1} + a_n = \frac{1}{2}\left(a_1 + a_2 + a_3 + \cdots + a_{n-2} + a_{n-1}\right)$$

$$\Rightarrow \frac{1}{2}a_{n-1} + a_n = \frac{1}{2}(a_1 + 2a_2)$$

$$\Rightarrow \frac{3l}{2} = \frac{1}{2}(a_1 + 2a_2)$$

$$\therefore \boxed{l = \frac{a_1 + 2a_2}{3}} \rightarrow \text{for A.P.}$$

For H.P

Change $l = 1/l$

$a_1 = 1/a_1$

$a_2 = 1/a_2$

$$\boxed{l = \frac{3a_1a_2}{2a_1 + a_2}}$$

Note:-

A.M.	G.M.
$a+b$	$a \cdot b$
$k \cdot a$	a^k

Note:- $a_{n+2} = \sqrt{a_n \cdot a_{n+1}}$, a_1 & a_2 are given then $\boxed{l = (a_1 a_2^2)^{1/3}}$

Example: ④:- $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$; a_1 is given. Discuss the convergence of the given sequence.

Solution:- $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$.

$$a_{n+1} - a_n = \frac{1}{2}a_n + \frac{1}{a_n} - a_n$$

$$= \frac{1}{a_n} - \frac{1}{2}a_n$$

$$= \frac{2 - a_n^2}{2a_n}$$

$$\Rightarrow 2a_{n+1}a_n = a_n^2 + 2$$

$$\Rightarrow a_n^2 - 2a_{n+1}a_n + 2 \geq 0$$

$$\Rightarrow 4a_{n+1}^2 - 4 \cdot 2 \geq 0$$

$$a_{n+1}^2 \geq 2$$

$$a_n^2 \geq 2 \Rightarrow a_n \geq \sqrt{2} \text{ or } a_n \leq -\sqrt{2}$$

Case (i): $a_1 > 1 \Rightarrow a_n > 0 \ \forall n$

$$a_{n+1} - a_n < 0 \quad \downarrow$$

Case (ii) :- $a_1 < 0$

$$a_{n+1} - a_n \geq 0 \quad \uparrow =$$

Example (5) :- $a_{n+2} = \sqrt{a_n a_{n+1}}$; $a_n > 0$

and a_1 & a_2 are given.

Show that $\langle a_n \rangle$ is convergent and find its limit.

Solution :- **Case (i) :-** If $a_1 = a_2$

$$a_3 = \sqrt{a_1 \cdot a_2} = a_1$$

$$a_4 = a_1$$

$\Rightarrow \langle a_n \rangle = \langle a_1, a_1, \dots \rangle$

$\Rightarrow \langle a_n \rangle$ is constant sequence

$\Rightarrow \langle a_n \rangle$ is convergent.

Case (ii) :- If $a_1 < a_2$

$$a_3 = \sqrt{a_1 a_2}$$

$$\Rightarrow a_1 < a_3 < a_2$$

$$a_4 = \sqrt{a_2 a_3}$$

$$a_1 < a_3 < a_4 < a_2$$

Similarly $a_1 < a_3 < a_5 < a_4 < a_2$

$$\Rightarrow a_1 < a_3 < a_5 < \dots < a_6 < a_4 < a_2$$

$\Rightarrow \langle a_{2r-1} \rangle = a_1 < a_3 < a_5 < \dots$ Lower bound $= a_1$

$\langle a_{2r} \rangle = a_2 > a_4 > a_6 > \dots$ Upper bound $= a_2$

$\Rightarrow$ Both $\langle a_{2r-1} \rangle$ and $\langle a_{2r} \rangle$ are bdd and monotonic.

$\Rightarrow \langle a_{2r-1} \rangle$ and $\langle a_{2r} \rangle$ are convergent.

Now, we have to show that limit of both sequences are same and hence given sequence is convergent.

Let $\langle a_{2r-1} \rangle \to l_1$ and $\langle a_{2r} \rangle \to l_2$

As $a_{n+2} = \sqrt{a_n a_{n+1}}$

$l_1 = \sqrt{l_1 l_2} \Rightarrow l_1^2 = l_1 l_2 \Rightarrow l_1 = l_2$

or $l_2 = \sqrt{l_2 l_1} \Rightarrow l_2^2 = l_2 l_1 \Rightarrow l_2 = l_1$ } $= l$ (say)

$\Rightarrow \lim \langle a_{2r-1} \rangle = \lim \langle a_{2r} \rangle = l$

$\Rightarrow \lim \langle a_n \rangle = l$

$\Rightarrow \langle a_n \rangle$ is convergent and converges to l.

Case(iii):- If $a_1 > a_2$ same as case(ii)

To find limit :-

$a_3 = \sqrt{a_1 a_2} \Rightarrow a_3^2 = a_1 a_2$

$a_4 = \sqrt{a_2 a_3} \Rightarrow a_4^2 = a_2 a_3$

$\vdots$

$a_{n+2} = \sqrt{a_n a_{n+1}} \Rightarrow (a_{n+2})^2 = a_n a_{n+1}$

Multiplying, we get

$a_3^2 \cdot a_4^2 \cdots a_n^2 a_{n+1}^2 a_{n+2}^2 = a_1 \cdot a_2^2 \cdot a_3^2 \cdots a_n^2 a_{n+1}$

$a_{n+1} a_{n+2}^2 = a_1 \cdot a_2^2$

Taking limit, we get

$l \cdot l^2 = a_1 \cdot a_2^2$

$\Rightarrow l^3 = a_1 \cdot a_2^2$

$\boxed{l = (a_1 \cdot a_2^2)^{1/3}}$

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Example (6):- If $\langle a_n \rangle$ & $\langle b_n \rangle$ are the sequences of +ve real numbers.

s.t. (i) $a_{n+1} = \sqrt{a_n b_n}$ (G.M); a_1, b_1 are given

(ii) $b_{n+1} = \frac{a_n + b_n}{2}$ (A.M); a_1, b_1 are given

then show that both the sequences are convergent and converges to the same limit.

Solution:- Since A.M $\geq$ G.M

$$b_{n+1} \geq a_{n+1} \quad \forall n$$

$$\Rightarrow \boxed{b_n \geq a_n} \Rightarrow a_1 \leq b_1$$

NOW $a_{n+1} = \sqrt{a_n b_n} \geq \sqrt{a_n a_n}$

$$a_{n+1} \geq \sqrt{a_n^2}$$

$$\Rightarrow \boxed{a_{n+1} \geq a_n} \Rightarrow \langle a_n \rangle \text{ is increasing sequence.}$$

Also $b_{n+1} = \frac{a_n + b_n}{2} \leq \frac{b_n + b_n}{2}$

$$\Rightarrow \boxed{b_{n+1} \leq b_n}$$

$\Rightarrow \langle b_n \rangle$ is decreasing sequence.

Since $a_n \leq a_{n+1}$

$$a_n \leq b_{n+1} \leq b_n \leq b_{n-1} \leq \cdots\cdots \leq b_1$$

$$a_n \leq b_1 \quad \forall n$$

$\Rightarrow \langle a_n \rangle$ is bdd above as well as monotonic.

$\Rightarrow \langle a_n \rangle$ is convergent.

And $b_{n+1} \leq b_n$

$$a_1 \leq a_2 \leq \cdots \leq a_{n+1} \leq b_n$$

$$a_1 \leq b_n \quad \forall n$$

$\Rightarrow \langle b_n \rangle$ is bounded below and monotonic.

$\Rightarrow \langle b_n \rangle$ is convergent.

Let $\langle a_n \rangle \to l_1$ and $\langle b_n \rangle \to l_2$

$$a_{n+1} = \sqrt{a_n b_n}$$

$$l_1 = \sqrt{l_1 l_2} \Rightarrow l_1^2 = l_1 l_2$$

$$\Rightarrow l_1 = l_2$$

& $b_{n+1} = \frac{a_n + b_n}{2}$

$$\Rightarrow l_2 = \frac{l_1 + l_2}{2}$$

$$\Rightarrow l_2 = l_1$$

Thus, both the sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ are convergent and converges to the same limit.

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Example ㉔:- If $0 < S_1 < S_2$ and $S_n = \frac{2S_{n-1}S_{n-2}}{S_{n-1} + S_{n-2}}$

then $\lim S_n = \frac{3S_1S_2}{2S_1 + S_2}$

Solution:- We have $S_n = \frac{2S_{n-1}S_{n-2}}{S_{n-1} + S_{n-2}}$

$$S_3 = \frac{2S_1S_2}{S_1 + S_2}$$

$$\Rightarrow \frac{1}{S_3} = \frac{S_1 + S_2}{2S_1S_2}$$

$$\Rightarrow \frac{1}{S_3} = \frac{1}{2}\left[\frac{1}{S_2} + \frac{1}{S_1}\right]$$ which is H.M

$$\frac{1}{S_4} = \frac{1}{2}\left[\frac{1}{S_3} + \frac{1}{S_2}\right]$$

$$\frac{1}{S_5} = \frac{1}{2}\left[\frac{1}{S_4} + \frac{1}{S_3}\right]$$

$$\vdots$$

$$\frac{1}{S_{n-1}} = \frac{1}{2}\left[\frac{1}{S_{n-2}} + \frac{1}{S_{n-3}}\right]$$

$$\frac{1}{S_n} = \frac{1}{2}\left[\frac{1}{S_{n-1}} + \frac{1}{S_{n-2}}\right]$$

Adding, we get

$$\cancel{\frac{1}{S_3}} + \cancel{\frac{1}{S_4}} + \cdots + \frac{1}{S_{n-2}} + \frac{1}{S_{n-1}} + \frac{1}{S_n} = \frac{1}{2S_1} + \frac{1}{S_2} + \cancel{\frac{1}{S_3}} + \cdots + \cancel{\frac{1}{S_{n-2}}} + \frac{1}{2S_{n-1}}$$

$$\frac{1}{2S_{n-1}} + \frac{1}{S_n} = \frac{1}{2S_1} + \frac{1}{S_2} \quad \text{---(1)}$$

Let 'l' be the limit of S_n

i.e. $\lim_{n\to\infty} S_n = l$ then from (1), we have

$$\frac{1}{2l} + \frac{1}{l} = \frac{S_2 + 2S_1}{2S_1S_2}$$

$$\Rightarrow \frac{3}{2l} = \frac{S_2 + 2S_1}{2S_1S_2}$$

$$\Rightarrow \boxed{l = \frac{3S_1S_2}{S_2 + 2S_1}}$$

$$\Rightarrow \boxed{\lim S_n = \frac{3S_1S_2}{S_2 + 2S_1}}$$ Hence proved

Cauchy Sequence :-

A sequence $\langle a_n \rangle$ is said to be Cauchy if for any $\varepsilon > 0$, $\exists\ p \in \mathbb{N}$ s.t.

$$|a_{n+p} - a_p| < \varepsilon \quad \forall\ n \in \mathbb{N}.$$

OR,

For any $\varepsilon > 0$, $\exists\ m_0 \in \mathbb{N}$

s.t. $|a_n - a_m| < \varepsilon \quad \forall\ n, m > m_0$.

OR,

For any $\varepsilon > 0$, $\exists\ m \in \mathbb{N}$

s.t. $|a_n - a_m| < \varepsilon \quad \forall\ n > m$.

Cauchy Criterian for convergence :-

A sequence is convergent iff it is Cauchy's sequence.

Proof :- Case (i) :- Here it is given that $\langle a_n \rangle$ is convergent and converges to l.

We shall prove $\langle a_n \rangle$ is a Cauchy's sequence.

Since $\langle a_n \rangle$ converges to l.

$\therefore$ For given $\varepsilon > 0$, $\exists$ a positive integer m s.t.

$$|a_n - l| < \varepsilon/2 \quad \forall\ n \geq m_1 \qquad \text{——— (1)}$$

or, $$|a_m - l| < \varepsilon/2 \quad \forall\ m \geq m_2 \qquad \text{——— (2)}$$

Take $m = \max\{m_1, m_2\}$

Now, $|a_n - a_m| = |a_n - l + l - a_m| \leq |a_n - l| + |l - a_m|$

$$= |a_n - l| + |a_m - l|$$

$$< \varepsilon/2 + \varepsilon/2 \quad \forall\ n \geq m$$

$$\Rightarrow |a_n - a_m| < \varepsilon \quad \forall\ n \geq m$$

Hence $\langle a_n \rangle$ is a Cauchy's sequence.

Case (ii) :- Converse

Here it is given that $\langle a_n \rangle$ is a Cauchy's sequence.
We shall prove that $\langle a_n \rangle$ is convergent.
Since $\langle a_n \rangle$ is a Cauchy's sequence.
$\therefore$ For given $\varepsilon > 0$, $\exists$ a +ve integer m such that

$$|a_n - a_m| < \varepsilon/3 \quad \forall\ n \geq m \quad \text{——— (3)}$$

Also every Cauchy's sequence is a bdd sequence.
$\therefore$ $\langle a_n \rangle$ is bdd. But every bdd sequence has a limit point l (say), therefore $\exists$ a +ve integer $k > m$ s.t.

$$|a_k - l| < \varepsilon/3 \quad \text{——— (4)}$$

Also (3) becomes

$$|a_k - a_m| < \varepsilon/3 \quad \text{——— (5)}$$

Now, $|a_n - l| = |a_n - a_m + a_m - a_k + a_k - l|$

$$\leq |a_n - a_m| + |a_m - a_k| + |a_k - l|$$

$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 \quad \forall\ n \geq m$$

[using (3), (4) & (5)]

$\therefore$ $|a_n - l| < \varepsilon \quad \forall\ n \geq m$.

Hence the sequence $\langle a_n \rangle$ converges to l.

Note (1) Cauchy criterian for convergence gives an important method to claim that sequence is not convergent.

i.e. $|a_{2n} - a_n| \geq c$ or $|a_{kn} - a_n| \geq c$

$\Rightarrow$ $\langle a_n \rangle$ is not convergent.

E.g. : If $a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$

To show that $\langle a_n \rangle$ is not convergent.

Solution:- $a_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n}$

$$|a_{2n} - a_n| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} \right|$$

$$\geq \left| \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n} \right|$$

$$\geq \left| n \cdot \frac{1}{2n} \right|$$

$$\geq \frac{1}{2}$$

If $\langle a_n \rangle$ is convergent then by Cauchy creteria for convergence for each $\varepsilon > 0$, $\exists\, m \in \mathbb{N}$ s.t

$$|a_n - a_m| < \varepsilon \quad \forall\, n > m$$

In particular, if we take $\varepsilon = 1/2$, we can not find $m \in \mathbb{N}$ s.t.

$$|a_{2n} - a_n| < 1/2 \quad \forall\, n > m.$$

Hence the sequence $\langle a_n \rangle$ is not convergent.

Eg:- If $a_n = 1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots + \frac{1}{4n-3}$

Show that $\langle a_n \rangle$ is not Cauchy.

Solution:- $a_{2n} = 1 + \frac{1}{5} + \frac{1}{9} + \frac{1}{13} + \cdots + \frac{1}{4n-3} + \frac{1}{4n+1} + \frac{1}{4n+5} + \cdots + \frac{1}{8n-3}$

$$|a_{2n} - a_n| = \left| \frac{1}{4n+1} + \frac{1}{4n+5} + \cdots + \frac{1}{8n-3} \right|$$

$$\Rightarrow |a_{2n} - a_n| \geq \frac{1}{8n} + \frac{1}{8n} + \cdots + \frac{1}{8n}$$

i.e $|a_{2n} - a_n| \geq \left| n \cdot \frac{1}{8n} \right|$

$\Rightarrow |a_{2n} - a_n| \geq \frac{1}{8}$

If we take $\varepsilon = \frac{1}{8}$, we can not find $m \in \mathbb{N}$ s.t.

$$|a_{2n} - a_n| < \frac{1}{8} \quad \forall\ n > m$$

Hence sequence $\langle a_n \rangle$ is not Cauchy sequence.

Eg: If $a_n = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1}$

Show that $\langle a_n \rangle$ is not Cauchy.

Solution:- $a_{2n} = 1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n-1} + \frac{1}{2n+1} + \cdots + \frac{1}{4n-1}$

$$|a_{2n} - a_n| = \left| \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-1} \right|$$

$$\geq \frac{1}{4n} + \frac{1}{4n} + \cdots + \frac{1}{4n}$$

$$\geq |n \cdot 1/4n|$$

$$\geq \frac{1}{4}$$

Hence the given sequence $\langle a_n \rangle$ is not Cauchy.

Note (2):- Construction of a sequence

Let $a \in \mathbb{R}$

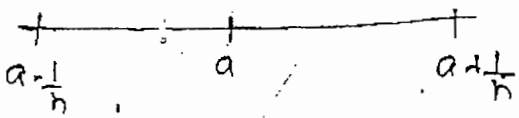
Define $a_n \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$

$a_1 \in (a-1, a+1)$

$a_2 \in \left(a - \frac{1}{2}, a + \frac{1}{2}\right)$

$\vdots$

$a_n \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$



For any $\varepsilon > 0$, $\exists\ m \in \mathbb{N}$ s.t. $\frac{1}{n} < \varepsilon$, $n > m$

$$|a_n - a| < \frac{1}{n} \quad \forall\ n \in \mathbb{N}$$

$$|a_n - a| < \frac{1}{n} < \varepsilon, \quad n > m$$

$\langle a_n \rangle \longrightarrow a$

Hence, we can construct a sequence of rationals only or sequence of irrationals only.

Cauchy First Theorem :→ If $\lim_{n\to\infty} a_n = 0$

then $\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0$

Hence if $\lim_{n\to\infty} a_n = l$

then $\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = l$ but not conversely.

Converse eg. $a_n = (-1)^n$

$b_n = \begin{cases} 0 & ; n \text{ is even} \\ 1/n & ; n \text{ is odd} \end{cases}$

$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{n} = 0$

but $\lim_{n\to\infty} a_n$ ✗

Proof :- $\lim_{n\to\infty} a_n = 0$

For any $\varepsilon > 0$, $\exists\ m_1$

$|a_n| < \varepsilon/2$ whenever $n > m_1$,

Also $|a_n| \leq M$

$$\left|\frac{a_1 + a_2 + \cdots + a_n}{n} - 0\right| = \frac{|a_1 + a_2 + \cdots + a_n|}{n}$$

$$\leq \frac{|a_1| + |a_2| + \cdots + |a_n|}{n}$$

$$< \frac{M + M + \cdots \text{ (} m_1 \text{ times)}}{n}$$

$$+ \frac{1}{2} \frac{(\varepsilon + \varepsilon + \cdots + (n - m_1) \text{ times})}{n}$$

$$< \frac{m_1 M}{n} + \frac{(n - m_1)}{n} \frac{\varepsilon}{2}$$

$$< \frac{m_1 M}{n} + \frac{\varepsilon}{2}$$

$$\left|\frac{a_1 + a_2 + \cdots + a_n}{n}\right| < \varepsilon/2 + \varepsilon/2 \quad \text{if} \quad \frac{m_1 M}{n} < \varepsilon/2$$

$$\left|\frac{a_1+a_2+\cdots+a_n}{n}\right| < \varepsilon \quad \text{if} \quad n > \frac{\varepsilon m_1 M}{\varepsilon}$$

$$\left|\frac{a_1+a_2+\cdots+a_n}{n}\right| < \varepsilon \quad \text{if} \quad n > \max\left\{m_1, \frac{\varepsilon m_1 M}{\varepsilon}\right\}$$

$$\Rightarrow \boxed{\lim_{n\to\infty} \frac{\sum a_n}{n} = 0}$$

When $\lim_{n\to\infty} a_n = l$ (given)

We have to prove

$$\lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} \to l$$

Set $b_n = a_n - l$

$b_1 = a_1 - l$

$b_2 = a_2 - l$

$\vdots$

$b_n = a_n - l$

Adding, we get :

$$b_1+b_2+\cdots+b_n = a_1+a_2+\cdots+a_n - nl$$

$$\frac{b_1+b_2+\cdots+b_n}{n} = \frac{a_1+a_2+\cdots+a_n}{n} - l$$

Taking limit on both sides, we get

$$\lim_{n\to\infty} \frac{b_1+b_2+\cdots+b_n}{n} = \lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} - l$$

Since $\langle a_n \rangle \to l$

$$\Rightarrow \langle b_n \rangle \to 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} \to l$$

Hence $\left\langle \left| \frac{\sum a_n}{n} \right| \right\rangle \to l$

Hence proved.

Cauchy II nd theorem on limits :- If $\langle a_n \rangle$ is a sequence such that $a_n > 0$ for all n and $\lim_{n\to\infty} a_n = a \neq 0$ then $\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = a$

Proof :- Let us define a sequence $\langle b_n \rangle$.

s.t. $b_n = \log a_n \quad \forall\, n$

Since $\lim_{n\to\infty} a_n = a$ then $\lim_{n\to\infty} b_n = \log a$

By Cauchy Ist theorem

$$\lim_{n\to\infty} \left(\frac{b_1 + b_2 + \cdots + b_n}{n} \right) = \log a$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{\log a_1 + \log a_2 + \cdots + \log a_n}{n} \right) = \log a$$

i.e $\lim_{n\to\infty} \log (a_1 a_2 \cdots a_n)^{1/n} = \log a$

and hence $\boxed{\lim_{n\to\infty} (a_1 a_2 \cdots a_n)^{1/n} = a}$

Hence proved.

Cauchy 2nd Theorem :- If $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$ exists finitely or infinitely then $(a_n)^{1/n}$ exists and $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$

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Cesaro Theorem :- If $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$ then

$$\lim_{n\to\infty} \frac{a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n}{n} = ab$$

Proof :- Suppose $a_n = c_n + a \Rightarrow \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n + a$

$\Rightarrow a = c_n + a$

$\Rightarrow c_n \to 0$

$a_n b_1 = c_n b_1 + a b_1$

$a_{n-1} b_2 = c_{n-1} b_2 + a b_2$

$\vdots \quad \vdots \quad \vdots$

$a_1 b_n = c_1 b_n + a b_n$

Adding we get,

$a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n = c_n b_1 + c_{n-1} b_2 + \cdots + c_1 b_n + a(b_1 + b_2 + \cdots + b_n)$

$\Rightarrow \dfrac{a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n}{n} = \dfrac{c_n b_1 + c_{n-1} b_2 + \cdots + c_1 b_n}{n} + a\left(\dfrac{b_1 + b_2 + \cdots + b_n}{n}\right)$

Taking limits on both sides, we have

$\lim_{n\to\infty} \dfrac{a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n}{n} = \lim_{n\to\infty} \dfrac{c_n b_1 + c_{n-1} b_2 + \cdots + c_1 b_n}{n} + \lim_{n\to\infty} a\left(\dfrac{b_1 + b_2 + \cdots + b_n}{n}\right)$

$\Rightarrow \lim_{n\to\infty} \dfrac{a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n}{n} = \lim_{n\to\infty} \dfrac{c_n b_1 + c_{n-1} b_2 + \cdots + c_1 b_n}{n} + ab$

[$\because$ By Cauchy first theorem $\lim_{n\to\infty} b_n \to b$

$\therefore \lim_{n\to\infty} \dfrac{b_1 + b_2 + \cdots + b_n}{n} = b$]

{As $b_n \to b \Rightarrow \langle b_n \rangle$ is convergent and hence bounded

$\Rightarrow |b_n| < M$ }

$\therefore \lim_{n\to\infty} \dfrac{a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n}{n} = \lim_{n\to\infty} \dfrac{Mc_n + \cdots + Mc_1 + ab}{n}$

$= \lim_{n\to\infty} \dfrac{M(c_1 + c_2 + \cdots + c_n)}{n} + ab$

$= M \lim_{n\to\infty} \dfrac{c_1 + c_2 + \cdots + c_n}{n} + ab$

$$\left[\text{As } c_n \to 0 \Rightarrow \frac{c_1+c_2+\cdots+c_n}{n} \to 0\right]$$

$$= M.0 + ab$$

$$= ab$$

$$\therefore \boxed{\lim_{n\to\infty} \frac{a_n b_1 + a_{n-1} b_2 + \cdots + a_1 b_n}{n} = ab}$$

Hence proved.

Algebra Of Limits :-

If $\lim a_n = a$, $\lim b_n = b$ then

(i) $\lim (a_n \pm b_n) = a \pm b$

(ii) $\lim (a_n b_n) = ab$

(iii) $\lim \left(\frac{a_n}{b_n}\right) = \frac{a}{b}$; $b \neq 0$, $b_n \neq 0$

(iv) $\lim |a_n| = |a|$ but not conversely e.g. $\langle a_n \rangle = (-1)^n$

(v) $\lim \max\{a_n, b_n\} = \max\{a, b\}$

(vi) $\lim \min\{a_n, b_n\} = \min\{a, b\}$

If $\lim a_n = a$, $\lim b_n = b$ then $\lim a_n b_n = ab$.

Proof :- Let $\epsilon > 0$ be any given number

Now, $|a_n b_n - ab| = |a_n b_n - a_n b + a_n b - ab|$

$= |a_n (b_n - b) + (a_n - a) b|$

$\leq |a_n| |b_n - b| + |a_n - a| |b| \quad \text{---} (1)$

Since sequence $\langle a_n \rangle$ is convergent, therefore $\langle a_n \rangle$ is bounded also.

$\therefore$ $\exists$ a number $k > 0$ such that $|a_n| < k$ $\forall$ n ∈ N $\quad \text{---} (2)$

Here sequence $\langle b_n \rangle$ is convergent to b (given)

$\therefore$ there exists a +ve integer m_1 s.t.

$$|b_n - b| < \frac{\varepsilon}{2k} \quad \forall\ n \geq m_1$$

Also for sequence $\langle a_n \rangle$ to be converges to 'a', there exists a positive integer m_2

$$\text{s.t. } |a_n - a| < \frac{\varepsilon}{2|b|} \quad \forall\ n \geq m_2$$

Let $m = \max\{m_1, m_2\}$

$\therefore |a_n - a| < \frac{\varepsilon}{2|b|}$ and $|b_n - b| < \frac{\varepsilon}{2k}$ $\forall\ n \geq m$ — (3)

Using (2) and (3) in (1), we get

$$|a_n b_n - ab| < k\left(\frac{\varepsilon}{2k}\right) + \frac{\varepsilon}{2|b|} \cdot |b| \quad \forall\ n \geq m$$

$$\Rightarrow |a_n b_n - ab| < \varepsilon \quad \forall\ n \geq m$$

$\therefore$ sequence $\langle a_n b_n \rangle$ converges to ab. Hence proved.

If $\lim a_n = a$, $\lim b_n = b$ then $\lim 1/a_n = 1/a$; $a \neq 0$, $a_n \neq 0$ $\forall\ n \in \mathbb{N}$

Proof:- Let $\varepsilon > 0$ be any given number.

Since $\langle a_n \rangle$ converges to 'a', therefore there exists a positive integer m_1 such that

$$|a_n - a| < \frac{|a|}{2} \quad \forall\ n \geq m_1 \text{ —— (1)}$$

Now, $|a| = |a - a_n + a_n|$

$$\leq |a - a_n| + |a_n|$$

$$= |a_n - a| + |a_n|$$

$$< \frac{|a|}{2} + |a_n|$$

$\therefore |a| - \frac{|a|}{2} < |a_n| \quad \forall\ n \geq m_1$

or, $\frac{|a|}{2} < |a_n| \quad \forall\ n \geq m_1$

or, $\frac{1}{|a_n|} < \frac{2}{|a|}$ $\forall\ n \geq m_1$ ——— (2)

Now, $\left|\frac{1}{a_n} - \frac{1}{a}\right| = \left|\frac{a - a_n}{a \cdot a_n}\right| = \frac{|a_n - a|}{|a||a_n|} = \frac{|a_n - a|}{|a|} \cdot \frac{1}{|a_n|}$

$< \frac{|a_n - a|}{|a|} \cdot \frac{2}{|a|}$ $\forall\ n \geq m_1$ [Using (2)]

$\Rightarrow \left|\frac{1}{a_n} - \frac{1}{a}\right| < 2\frac{|a_n - a|}{|a|^2}$ $\forall\ n \geq m_1$ ——— (3)

As $a_n \to a$, therefore there exists a +ve integer m_2 s.t. $|a_n - a| < \varepsilon \frac{|a|^2}{2}$ $\forall\ n \geq m_2$ ——— (4)

Let $m = \max\{m_1, m_2\}$

From (3) and (4), we get

$$\left|\frac{1}{a_n} - \frac{1}{a}\right| < \frac{2}{|a|^2}|a_n - a| < \frac{2}{|a|^2} \cdot \frac{\varepsilon |a|^2}{2} \quad \forall\ n \geq m.$$

or, $\left|\frac{1}{a_n} - \frac{1}{a}\right| < \varepsilon$ $\forall\ n \geq m$

$\therefore$ sequence $\left\langle \frac{1}{a_n} \right\rangle$ converges to $\frac{1}{a}$. Hence proved.

If $\lim a_n = a$, $\lim b_n = b$ then
$\lim \frac{a_n}{b_n} = \frac{a}{b}$, $b \neq 0$, $b_n \neq 0$ $\forall\ n \in \mathbb{N}$

Proof:- As $b_n \neq 0$ for any n and $\langle b_n \rangle$ converges to $b \neq 0$

$\therefore \lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$

Now, $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left(a_n \cdot \frac{1}{b_n}\right)$

$= \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} \frac{1}{b_n} = a \cdot \frac{1}{b}$

$\therefore$ Sequence $\left\langle \frac{a_n}{b_n} \right\rangle$ converges to $\frac{a}{b}$

Hence proved.

If $\lim a_n = a$, $\lim b_n = b$ then $\lim (a_n + b_n) = a + b$

Proof:- Let $\varepsilon > 0$ be given number

Since $\langle a_n \rangle$ converges to 'a' and $\langle b_n \rangle$ converges to 'b', therefore $\exists$ +ve integers m_1 & m_2 s.t.

$|a_n - a| < \varepsilon/2 \ \forall\ n \geq m_1$ & $|b_n - b| < \varepsilon/2 \ \forall\ n \geq m_2$

Let $m = \max\{m_1, m_2\}$

$\therefore |a_n - a| < \varepsilon/2$ and $|b_n - b| < \varepsilon/2 \ \forall\ n \geq m$ ——— (1)

Now, $|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)|$

$\leq |a_n - a| + |b_n - b|$

$< \varepsilon/2 + \varepsilon/2$ [Using (1)]

Thus we have $|(a_n + b_n) - (a + b)| < \varepsilon \ \forall\ n \geq m$

Hence $\langle a_n + b_n \rangle$ converges to $a + b$.

Hence proved

Diverging to $+\infty$:- A sequence is said to diverge to $+\infty$ if for any $M > 0$, $\exists\ m \in \mathbb{N}$ s.t. $a_n > M \ \forall\ n \geq m$

i.e. $\lim_{n \to \infty} a_n \longrightarrow +\infty$

e.g.: $a_n = n$.

Diverging to $-\infty$:- A sequence is said to diverge to $-\infty$ if for any $M > 0$, $\exists\ m \in \mathbb{N}$ s.t. $a_n < -M \ \forall\ n \geq m$

i.e. $\lim_{n \to \infty} a_n \longrightarrow -\infty$

e.g. $a_n = -n$

Oscillatory Sequence:- A sequence $\langle a_n \rangle$ is said to be an oscillatory sequence if it is neither convergent nor divergent.

An oscillatory sequence is said to oscillate finitely or infinitely according as it is bounded or unbounded.

eg (i) $\langle (-1)^n \rangle \longrightarrow$ oscillate finitely

(ii) $\langle (-1)^n n \rangle \longrightarrow$ oscillate infinitely

Limit Superior & Limit Inferior

Limit Superior:- Let $\langle a_n \rangle$ be a given sequence

Define $M_n = \sup_n \{ a_n, a_{n+1}, a_{n+2}, \cdots \}$

$M_{n+1} = \sup_n \{ a_{n+1}, a_{n+2}, \cdots \}$ (B.S.A)

Now, $M_n \geq M_{n+1} \Rightarrow \langle M_n \rangle \downarrow$

$\rightarrow$ If $\langle M_n \rangle$ is convergent $\Rightarrow \langle M_n \rangle \to \inf$.

and this limit is called <u>limit superior</u>.

$$\lim_{n\to\infty} M_n = \inf \{ M_1, M_2, \cdots \} = \limsup \langle a_n \rangle.$$

Limit inferior:- Let $\langle a_n \rangle$ be a given sequence.

Define $m_n = \inf_n \{ a_n, a_{n+1}, a_{n+2}, \cdots \}$

$m_{n+1} = \inf_n \{ a_{n+1}, a_{n+2}, \cdots \}$

$\Rightarrow m_n \leq m_{n+1}$

$\Rightarrow$ Sequence $\langle m_n \rangle$ is increasing sequence.

If $\langle m_n \rangle$ is convergent $\Rightarrow \langle m_n \rangle \to \sup$

and this limit is called limit inferior.

$$\lim_{n\to\infty} m_n = \sup \{ m_1, m_2, \cdots \} = \liminf \langle a_n \rangle.$$

Result:- Let $\langle a_n \rangle$ be a <u>bdd</u> sequence and A be the set of all limit points of the sequence $\langle a_n \rangle$. then

$\limsup (a_n) = \sup A$

$\liminf (a_n) = \inf A.$

Problem:- (i) $a_n = \left(\cos \frac{n\pi}{4}\right)^{(-1)^n}$

(ii) $a_n = \sin n\frac{\pi}{2} + (-1)^n$

find limit sup and lim inf.

Solution:- (i) $a_n = \left(\cos \frac{n\pi}{4}\right)^{(-1)^n}$

$= \left\{\sqrt{2}, 0, -\sqrt{2}, -1, \cdots\right\}$

lim sup $\langle a_n \rangle = \sqrt{2}$

lim inf $\langle a_n \rangle = -\sqrt{2}$

(ii) $a_n = \sin \frac{n\pi}{2} + (-1)^n$

$= \{0, 1, -2, 1, \cdots\}$

lim sup $\langle a_n \rangle = 1$, lim inf $= -2$

→ Let $\langle a_n \rangle = \langle (-1)^n \rangle = \langle -1, 1, -1, 1, \cdots \rangle$

Set of limit points $= \langle -1, 1 \rangle$

lim sup $\langle a_n \rangle = 1$, lim inf $\langle a_n \rangle = -1$

→ Let $\langle a_n \rangle = \langle 1 + (-1)^n \rangle$

$= \langle 0, 2, 0, 2, \cdots \rangle$

Set of limit points $= \langle 0, 2 \rangle$

lim sup $\langle a_n \rangle = 2$, lim inf $\langle a_n \rangle = 0$

→ Let $\langle a_n \rangle = \langle 1/n \rangle = \langle 1, 1/2, 1/3, 1/4, \cdots \rangle$

It has exactly one limit point i.e. 0

lim sup $\langle a_n \rangle = 0$, lim inf $\langle a_n \rangle = 0$.

→ Let $\langle a_n \rangle = \langle (-1)^n \left(1 + \frac{1}{n}\right) \rangle$

$= \langle -2, 1+\frac{1}{2}, -\left(1+\frac{1}{3}\right), \left(1+\frac{1}{4}\right), \cdots \rangle$

$= \langle -2, 3/2, -4/3, 5/4, \cdots \rangle$

Set of limit points $= \langle -1, 1 \rangle$

$\liminf \langle a_n \rangle = -1, \quad \limsup \langle a_n \rangle = 1$

→ Let $\langle a_n \rangle = \langle n(1+(-1)^n) \rangle$

$= \langle 0, 4, 0, 8, 0, 12, 0, \cdots \rangle$

The sequence is bdd below but not bdd above.

Its limit point is 0.

$\limsup \langle a_n \rangle = +\infty$, $\liminf \langle a_n \rangle = 0$.

→ Let $\langle a_n \rangle = \langle 2, 4, 2, 4, 2, 4, \cdots \rangle$

Set of limit points $= \langle 2, 4 \rangle$.

$\liminf \langle a_n \rangle = 2, \quad \limsup \langle a_n \rangle = 4$

→ Let $\langle a_n \rangle = \langle 1/2n \rangle = \langle 1/2, 1/4, 1/6, 1/8, \cdots \rangle$

0 is the only limit point of $\langle 1/2n \rangle$

$\limsup \langle a_n \rangle = 0, \quad \liminf \langle a_n \rangle = 0.$

→ Let $\langle a_n \rangle = \langle 1, 3, 5, 1, 3, 5, 1, 3, 5, \cdots \rangle$

Set of limit points $= \langle 1, 3, 5 \rangle$

$\limsup \langle a_n \rangle = 5, \quad \liminf \langle a_n \rangle = 1.$

→ Let $\langle a_n \rangle = \langle (-2)^n (1+1/n) : n \in \mathbb{N} \rangle$

$= \langle -2(1+1), 4(1+\frac{1}{2}), -8(1+\frac{1}{3}), \cdots \rangle$

$\langle a_n \rangle$ is neither bdd above nor bdd below

$\limsup \langle a_n \rangle = +\infty, \quad \liminf \langle a_n \rangle = -\infty.$

→ $\langle (-1)^n n \rangle = \langle -1, 2, -3, 4, -5, 6, \text{---} \rangle = \langle a_n \rangle$

$\langle a_n \rangle$ is neither bold above nor bold below.

$\limsup \langle a_n \rangle = +\infty$, $\liminf \langle a_n \rangle = -\infty$.

→ $\langle a_n \rangle = \langle \sin n\frac{\pi}{3} \rangle = \langle \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, -\frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0, \text{---} \rangle$

Set of limit points $= \{ \frac{\sqrt{3}}{2}, -\frac{\sqrt{3}}{2}, 0 \}$

$\limsup \langle a_n \rangle = \frac{\sqrt{3}}{2}$, $\liminf \langle a_n \rangle = -\frac{\sqrt{3}}{2}$.

→ $\langle a_n \rangle = \langle 1 + \frac{(-1)^n}{n} \rangle = \langle 0, 1+\frac{1}{2}, 1-\frac{1}{3}, 1+\frac{1}{4}, \text{---} \rangle$

Set of limit point $= \langle 1 \rangle$

$\boxed{\limsup \langle a_n \rangle = 1 = \liminf \langle a_n \rangle}$

V.Imp →

$\langle a_n \rangle = \langle (1+\frac{1}{n})^{n+1} \rangle$

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (1+\frac{1}{n})^{n+1} = e$

$\boxed{\limsup \langle a_n \rangle = \liminf \langle a_n \rangle = e}$

→ $\langle a_n \rangle = \langle \frac{(-1)^n}{n^2} \rangle$

$\lim_{n\to\infty} a_n = 0$

$\boxed{\limsup \langle a_n \rangle = \liminf \langle a_n \rangle = 0}$

→ $S = \{ x + \frac{1}{n} : n \in \mathbb{N} \ \& \ x \in (a, b) \}$

$S' = [a, b]$

$\limsup \langle a_n \rangle = \lim_{n\to\infty} M_n$

$M_n = \sup(M_n, M_{n+1}, \dots)$

$M_n \geq M_{n+1}$

$\limsup \langle a_n \rangle = \inf(M_1, M_2, \dots M_n, \dots)$

$\xrightarrow{\text{V.gmp}} \langle a_n \rangle = \langle \sin 1/n \rangle = \langle \sin 1, \sin 1/2, \sin 1/3, \dots \rangle$

$\boxed{\limsup \langle a_n \rangle = \liminf \langle a_n \rangle = 0}$

Result:- When sequence is bounded and it has unique limit point, then it is convergent and

$\limsup \langle a_n \rangle = \liminf \langle a_n \rangle$

Hence if sequence is bounded and $\limsup = \liminf$

$\Rightarrow$ sequence is convergent.

$\rightarrow$ $\langle a_n \rangle$ s.t. $a_n = n$

$\langle a_n \rangle$ is increasing sequence

$M_1 = \sup\{1, 2, 3, \dots\} = \infty$

$M_2 = \sup\{2, 3, \dots\} = \infty$

$\limsup a_n = \inf\{\infty, \infty, \dots\} = \infty$

$m_1 = \inf\{1, 2, 3, \dots\} = 1$

$m_2 = \inf\{2, 3, \dots\} = 2$

$\liminf a_n = \sup\{1, 2, 3, \dots\} = \infty$

$\therefore$ $\boxed{\limsup a_n = \liminf a_n = \infty}$

Result:- (i) If $\langle a_n \rangle$ diverges to $+\infty$ or $-\infty$ then $\limsup = \liminf = +\infty$ or $-\infty$ and conversely.

(ii) If $\langle a_n \rangle$ is bdd then

$S = \{l \in \mathbb{R} : l \text{ is a l.p. of } \langle a_n \rangle\}$

S is non-empty.

$$\boxed{\begin{aligned}\limsup a_n &= \sup S \\ \liminf a_n &= \inf S\end{aligned}}$$

→ $\langle a_n \rangle$ s.t. $a_n = \begin{cases} 0 & ; n \text{ is prime or } 1 \\ p & ; n \text{ is composite and } p \text{ is the smallest prime dividing } n. \end{cases}$

Limit point of $\langle a_n \rangle = p \cup \{0\}$

$\langle a_n \rangle = \{0, 0, 0, 2, 0, 2 \sim\sim\}$

$M_1 = \sup\{0, 0, 0, 2, 0, 2 \cdots\} = \infty$

$M_2 = \sup\{0, 0, 2, 0, 2 \sim\sim\} = \infty$

Lim sup $a_n = \inf\{\infty, \infty, \cdots\} = \infty$.

$m_1 = \inf\{0, 0, 0, 2, 0, 2 \sim\sim\} = 0$

$m_2 = \inf\{0, 0, 2, 0, 2 \sim\sim\} = 0$

Lim inf $a_n = \sup\{0, 0, 0, \sim\sim\} = 0$

$$\boxed{\begin{aligned}\limsup a_n &= \infty \\ \liminf a_n &= 0\end{aligned}}$$

→ Bounded sequence never diverge.

→ Unbounded sequence never converge. It is always divergent or oscillates infinitely.

→ $\langle a_n \rangle = -2, -1, 0, 1, 2, 3, 1, 2, 1, 2, \cdots$

$$\left.\begin{aligned}\limsup a_n &= 2 \\ \liminf a_n &= 1 \\ \sup a_n &= 3 \\ \inf a_n &= -2\end{aligned}\right\} \text{All are different.}$$

→ If a sequence has more than one limit point then sequence must be oscillatory.

→ A sequence is convergent iff lim sup = lim inf

→ Null sequence :- $|a_n| < \varepsilon \ \forall \ n \geq m_0$

OR, $\langle a_n \rangle$ is called null sequence if $\lim \langle a_n \rangle = 0$

→ $\langle a_n \rangle$ is not a null sequence and not convergent then $\langle a_n \rangle = (-1)^n a_n$ is oscillating sequence

→ $\langle \sin (n+1) \frac{\pi}{2} \rangle$ is bdd sequence

$$\boxed{\begin{aligned} &\text{lim sup} = 1 \\ &\text{lim inf} = -1 \end{aligned}}$$

→ $\cos \frac{1}{n}$ is converges to $\cos \frac{1}{\infty} = \cos 0 = 1$

$$\therefore \boxed{\begin{aligned} &\text{lim sup} = 1 \\ &\text{lim inf} = 1 \end{aligned}}$$

→ $\langle a_n \rangle = (-1)^n n^2$ is oscillating infinitely

$\langle a_n \rangle = \{-1, 4, -9, 16, -25, \cdots\}$

$$\boxed{\begin{aligned} &\text{lim sup } a_n = \infty \\ &\text{lim inf} = -\infty \end{aligned}}$$

$\overline{\lim}\, a_n = b$, $\underline{\lim}\, a_n = a$

Result:- A sequence is said to be divergent in any of the following cases :-

(i) $\overline{\lim}\, a_n = \underline{\lim}\, a_n = +\infty$

(ii) $\overline{\lim}\, a_n = \underline{\lim}\, a_n = -\infty$

e.g. (a) $a_n = n$ it diverges to $+\infty$

(b) $a_n = -n$ it diverges to $-\infty$

(iii) The sequence is said to be oscillatory if $\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$

Note:- If $\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$, if either of them is $+\infty$ or $-\infty$ then sequence oscillates infinitely.

(iv) If $\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$ but both are finite then sequence oscillates finitely.

(v) A bdd sequence oscillates finitely if $\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$.

(vi) A sequence is convergent iff lim sup and lim inf exists finitely and equal.

(vii) If $\langle a_n \rangle$ is bounded then

$\overline{\lim}\, a_n = \sup\{\text{set of l.p. of } \langle a_n \rangle\} = \beta$

$\underline{\lim}\, a_n = \inf\{\text{set of l.p. of } \langle a_n \rangle\} = \alpha$

then $a_n \notin (\alpha, \beta)$ if $\alpha \neq \beta$.

for only finite values of n.

→ If $\langle a_n \rangle \to l$ then $f(\langle a_n \rangle) \to f(l)$; f is continuous at $x = l$.

→ $a_n \to l$

$\lim a^{a_n} = a^{\lim a_n} = a^l$

→ $\frac{\sum n}{n^2} = \frac{n(n+1)}{2n^2} = \frac{n+1}{2n} = \frac{1}{2} + \frac{1}{2n}$

$\lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2n}\right) = 1/2$

→ $a_n = \sin \pi \sqrt{n^2+1}$

$= \sin n\pi \left(1 + \frac{1}{n^2}\right)^{1/2}$

$= \sin n\pi \left(1 + \frac{1}{2n^2} + \cdots\right)$

$= \sin\left(n\pi + \frac{n\pi}{2n^2} + \cdots\right)$

$\therefore \boxed{\lim_{n\to\infty} \sin \pi \sqrt{n^2+1} = 0}$

→ $\sin\left[(-1)^n \frac{n\pi}{2}\right] + \cos \frac{n\pi}{2}$

$a_1 = -1$

$a_2 = -1$

Range $\langle a_n \rangle = \{-1, 1\}$

$\boxed{\overline{\lim}\, a_n = 1, \quad \underline{\lim}\, a_n = -1}$

Here $\overline{\lim}\, a_n \neq \underline{\lim}\, a_n$ then the sequence $\langle a_n \rangle$ oscillates finitely.

→ $\langle a_n \rangle = \sin n$

$\boxed{\overline{\lim}\, a_n = 1, \quad \underline{\lim}\, a_n = -1}$

→ $a_n = \log_e(n+1)$

$\lim a_n = +\infty$, $\underline{\lim}\, a_n = +\infty$

$\sup \langle a_n \rangle = \infty$, $\inf \langle a_n \rangle = \log_e 2$

Results:- (i) lim sup or lim inf of a monotonic sequences are always equal and if bdd then it is equal to the limit of the sequence and if unbounded then $+\infty$ in case of increasing and $-\infty$ in case of decreasing.

(ii) If sequence $\langle a_n \rangle$ is bounded and a, b are lim inf and lim sup respectively then only finite members of the sequence may lie outside the $[a, b]$.

Series

Definition:- Let $\langle a_n \rangle$ be a given sequence of real numbers then $\sum a_n$ is defined as series of real numbers.
$\sum a_n$ is convergent if its sum can be obtained fixed and finitely.

Define S_n : Set $S_n = \sum_{r=1}^{n} a_r$

$$S_1 = a_1$$
$$S_2 = a_1 + a_2$$
$$\vdots$$
$$S_n = a_1 + a_2 + \cdots + a_n \text{ is the } n\text{th}$$

partial sum of $\sum a_n$.
$\langle S_n \rangle$ denote the sequence of partial sum (SOPS)

→ $\sum a_n$ is convergent iff $\langle S_n \rangle$ is convergent.
$\langle S_n \rangle$ is convergent iff $\langle S_n \rangle$ is Cauchy.
i.e. $|S_{n+p} - S_p| < \varepsilon \quad \forall\, n \in \mathbb{N}$

or, $|S_{p+1} - S_p| < \varepsilon$

$|a_{n+1}| < \varepsilon$

$\Rightarrow \boxed{\lim a_n = 0}$

Note: (i) If $\sum a_n$ is cgt $\Rightarrow \lim a_n = 0$ (i.e. $\langle a_n \rangle$ is null sequence)
i.e. if a series is convergent then its limit is zero (necessary) but not sufficient.

(ii) $\lim a_n = 0 \not\Rightarrow \sum a_n$ is convergent.

Problem:- If $a_n = 1/n$, then show that $\sum a_n$ is not convergent.

Solution:- $\sum a_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots$

$$S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$S_{2n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$$

$$|S_{2n} - S_n| = \left| \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right|$$

$$\Rightarrow \frac{1}{2n} + \frac{1}{2n} + \cdots + \frac{1}{2n}$$

$$\Rightarrow |S_{2n} - S_n| > 1/2$$

If we take $\varepsilon = 1/2$, we cannot find $m \in \mathbb{N}$ s.t. $|S_{2n} - S_n| < \frac{1}{2}$ $\forall$ $n > m$

Hence $\langle S_n \rangle$ is not Cauchy

Hence $\sum a_n$ is not convergent.

Note:- (i) $\sum a_n$ is convergent →
- $\langle a_n \rangle$ is sgt and converges to 0
- $\langle S_n \rangle$ is convergent and converges to the value of the series need not to be 0.

(ii) $\sum a_n$ is divergent →
- $\langle a_n \rangle$ cannot say
  $\langle a_n \rangle = 1/n \to$ sgt
  $\langle a_n \rangle = n \to$ dgt
- $\langle S_n \rangle$ is divergent.

Telescopic Series :- Let $\langle a_n \rangle$ be a sequence of real numbers.

Define $b_n = a_{n+1} - a_n \Rightarrow S_n = a_{n+1} - a_1$

$c_n = a_n - a_{n+1} \Rightarrow T_n = a_1 - a_{n+1}$

then Σb_n and Σc_n are called telescopic series.

i.e. Let $\langle a_n \rangle$ be a sequence of real numbers and $b_n = a_{n+1} - a_n$ then Σb_n is called telescopic series.

$S_1 = b_1$

$S_1 = a_2 - a_1$

$S_2 = b_1 + b_2$

$= a_2 - a_1 + a_3 - a_2$

$S_2 = a_3 - a_1$

$\vdots$

$S_n = a_{n+1} - a_1$

$\lim S_n = \lim (a_{n+1} - a_1) = \lim a_{n+1} - a_1$.

Hence Σb_n is convergent iff sequence $\langle a_n \rangle$ is cgt.

Properties :- The telescopic series Σb_n and Σc_n are convergent iff $\langle S_n \rangle$ and $\langle T_n \rangle$ are convergent and converges to $l - a_1$ and $a_1 - l$ respectively.

Where $\lim a_n = l$.

e.g. :- $\sum \frac{1}{n(n+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100} + \frac{1}{100\cdot 101} + \cdots$

Solution :- $b_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1} \Rightarrow b_1 = 1 - \frac{1}{2} = S_1$

$$S_2 = b_1 + b_2 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3}$$

$$\boxed{S_n = 1 - \frac{1}{n+1}}$$

$$\boxed{\lim S_n = 1}$$

$$\boxed{\sum b_n = 1.}$$

e.g. :- $a_n = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{99\cdot 100}$

$$\lim S_n = 1 - \frac{1}{100} = \frac{99}{100} \text{ Ans}$$

Another method

$$a_n = 1 - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - \cancel{\frac{1}{3}} + \cdots + \cancel{\frac{1}{99}} - \frac{1}{100}$$

$$= 1 - \frac{1}{100} = \frac{99}{100} \text{ Ans}$$

e.g. :- $\sum a_n = \sum \left(1 - \frac{(-1)^n}{n}\right)$

Define $b_n = a_{n+1} - a_n$

$$S_1 = b_1 = a_2 - a_1$$

$$S_2 = b_1 + b_2 = \cancel{a_2} - a_1 + a_3 - \cancel{a_2} = a_3 - a_1$$

$$\vdots$$

$$S_n = b_1 + b_2 + \cdots + b_n = a_{n+1} - a_1$$

$$\lim S_n = \lim (a_{n+1} - a_1) = \lim a_{n+1} - a_1$$

$$\lim s_n = \lim \left[1 - \frac{(-1)^n}{n}\right] - a_1$$

$$= 1 - 2$$

$$\lim s_n = -1$$

$$\boxed{\sum a_n = -1}$$

Problem:- If $\sum a_n$ is convergent and $b_n = a_{n+1} - a_n$ then $\sum b_n$ is convergent.

Solution:- Since $\sum a_n$ is convergent $\Rightarrow \lim a_n = 0$

$$b_1 = a_2 - a_1$$
$$b_2 = a_3 - a_2$$
$$\vdots$$
$$b_n = a_{n+1} - a_n$$

$$s_1 = b_1 = a_2 - a_1$$
$$s_2 = b_1 + b_2 = a_2 - a_1 + a_3 - a_2 = a_3 - a_1$$
$$\vdots$$
$$s_n = a_{n+1} - a_1$$

$$\lim s_n = \lim (a_{n+1} - a_1)$$

$\Rightarrow \lim s_n = 0 - a_1 = -a_1 \rightarrow$ Finite $\Rightarrow$ convergent.

Thus $\sum b_n$ is convergent.

Test:-

1). If $\langle a_n \rangle$ and $\langle b_n \rangle$ are sequences of positive terms such that $\lim \frac{a_n}{b_n}$ exists finitely and non-zero then $\sum a_n$ & $\sum b_n$ behaves alike.

2). If $a_n \geq b_n$ then $\sum a_n$ is convergent $\Rightarrow \sum b_n$ is cgt. and $\sum b_n$ is divergent $\Rightarrow \sum a_n$ is divergent.

3). If $\lim \frac{a_n}{b_n} = 0$; If $\sum b_n$ is convergent

$\Rightarrow \sum a_n$ is convergent.

4). If $\lim \frac{a_n}{b_n} = \infty$

If $\sum b_n$ is divergent $\Rightarrow \sum a_n$ is divergent

e.g.:- $\frac{a_{n+1}}{b_{n+1}} < \frac{a_n}{b_n}$

$\Rightarrow a_{n+1} b_n < a_n b_{n+1}$

$a_2 b_1 < a_1 b_2$

$a_3 b_2 < a_2 b_3$

$\vdots$

$a_{n+1} b_n < a_n b_{n+1}$

Multiplying we get

$b_1 b_2 \cdots b_{n-1} a_2 a_3 \cdots a_n < a_1 a_2 \cdots a_{n-1} b_2 b_3 \cdots b_n$

$b_1 a_n < a_1 b_n$

$\frac{a_n}{a_1} < \frac{b_n}{b_1}$

$a_n < \frac{a_1}{b_1} b_n$

$a_n < k b_n$ where $k = \frac{a_1}{b_1}$

$\Rightarrow$ If $\sum b_n$ is convergent $\Rightarrow \sum a_n$ is convergent.

& if $\sum a_n$ is divergent $\Rightarrow \sum b_n$ is divergent.

Problem:- Find the supremum and the infimum of the given sequence $\langle n^2 + \frac{1}{n^2} + 3 \rangle$.

Solution:- $\langle n^2 + \frac{1}{n^2} + 3 \rangle = \{5, \frac{29}{4}, \cdots\}$

$g.l.b = 5$

supremum does not exist.

Leibnitz Test:-

If alternating series $\sum (-1)^n a_n$, $a_n > 0$ satisfies

(i) $a_{n+1} \leq a_n$ i.e. given series must be $\downarrow$

(ii) $\lim a_n = 0$

then $\sum (-1)^n a_n$ converges in its form

i.e. by form we mean the rearrangement of terms can give the different series or another form.

e.g.:- $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ is convergent.

Comparision Test in the Limit Form:- If $\sum a_n$ & $\sum b_n$ are series of +ve real numbers

if $\lim_{n \to \infty} \frac{a_n}{b_n} = l \neq 0 \neq \infty$ then $\sum a_n$ & $\sum b_n$ converge or diverge together.

Note:- Removable of finite number of terms from a series of +ve real numbers does not affect the convergence or divergence.

Important :- $\sum \frac{1}{n^p}$ is convergent if $p > 1$
divergent if $p \leq 1$

e.g :- $\sum \sin \frac{1}{n}$

$a_n = \sin 1/n$

$b_n = 1/n$

But $\sum b_n$ is divergent

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sin 1/n}{1/n} = 1$$

$\therefore$ **$\sum \sin 1/n$ is divergent**

e.g :- $\sum \sin 1/n^2$

$a_n = \sin 1/n^2$

$b_n = 1/n^2$

But $\sum b_n$ is convergent

$$\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\sin 1/n^2}{1/n^2} = 1$$

$\therefore$ **$\sum \sin 1/n^2$ is convergent**

e.g :- $a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1} + \sqrt{n}}$

$b_n = \frac{1}{\sqrt{n}}$

$\therefore$ **$\sum (\sqrt{n+1} - \sqrt{n})$ is divergent**

Results:- (i) If Σa_n is convergent and $a_n > 0$ then $\Sigma (a_n)^2$ is convergent but not conversely.

e.g. $\Sigma 1/n^2$ is convergent but $\Sigma 1/n$ is divergent.

(ii) If Σa_n is convergent and $a_n > 0$ then $\Sigma \frac{a_n}{1+a_n}$ is convergent, $\Sigma \frac{a_n}{1-a_n}$ is convergent; $a_n \neq 1$.

Proof:- Since $\frac{a_n}{1+a_n} < a_n$

$\Rightarrow \Sigma \frac{a_n}{1+a_n} < \Sigma a_n \to$ cgt

Hence by comparision test

$\Sigma \frac{a_n}{1+a_n}$ is convergent.

And $\Sigma \frac{a_n}{1-a_n} = \Sigma u_n$

$\Sigma v_n = \Sigma a_n$

$\lim_{n\to\infty} \frac{v_n}{u_n} = \lim \frac{a_n}{a_n/1-a_n} = \lim (1-a_n) = 1 - \lim a_n$

$= 1$ (Finite)

$\downarrow$ cgt.

$\therefore \Sigma \frac{a_n}{1-a_n}$ is convergent.

(iii) If Σa_n^2 & Σb_n^2 are convergent series of +ve real numbers then $\Sigma \sqrt{a_n b_n}$ is convergent.

Proof:- As $(a_n - b_n)^2 \geq 0 \Rightarrow a_n^2 + b_n^2 - 2a_n b_n \geq 0$

$\Rightarrow \Sigma a_n^2 + \Sigma b_n^2 \geq \Sigma 2\sqrt{a_n b_n}$

$\Rightarrow \Sigma a_n^2 + \Sigma b_n^2 \geq \Sigma \sqrt{a_n b_n}$

By comparision test,

$\Sigma \sqrt{a_n b_n}$ is convergent

Hence proved

(iv). If $\sum a_n$ is convergent then $\sum \frac{a_n}{n}$ is convergent.

Proof :- As $\left(a_n - \frac{1}{n}\right)^2 \geq 0$

$\Rightarrow a_n^2 + \frac{1}{n^2} - \frac{2a_n}{n} \geq 0$

$\Rightarrow \sum a_n^2 + \sum \frac{1}{n^2} \geq \sum \frac{2a_n}{n}$

$\sum a_n^2$ cgt, $\sum \frac{1}{n^2}$ cgt

$\Rightarrow \sum a_n^2 + \sum \frac{1}{n^2}$ is convergent

Then by comparision test $\sum \frac{2a_n}{n}$ is convergent

$\Rightarrow \sum \frac{a_n}{n}$ is convergent.

Hence proved.

(v). If $\sum a_n$ is convergent and $a_n > 0$ then $\sum \frac{\sqrt{a_n}}{n}$ is cgt.

Proof :- As $\left(\sqrt{a_n} - \frac{1}{n}\right)^2 \geq 0$

$\Rightarrow a_n + \frac{1}{n^2} - \frac{2\sqrt{a_n}}{n} \geq 0$

$\Rightarrow \sum \left(a_n + \frac{1}{n^2}\right) \geq \sum \frac{2\sqrt{a_n}}{n}$

cgt, cgt

$\Rightarrow \sum \left(a_n + \frac{1}{n^2}\right)$ is convergent

Then, by comparison test, $\sum \frac{2\sqrt{a_n}}{n}$ is convergent

$\Rightarrow \frac{\sqrt{a_n}}{n}$ is convergent

Hence proved

Algebra of Series :-

(i) If Σa_n and Σb_n are convergent then $\Sigma(a_n + b_n)$ and $\Sigma(a_n - b_n)$ is convergent. In fact any linear combination of convergent series is convergent i.e. $\Sigma(\lambda a_n \pm \mu b_n)$ is convergent.

(ii) If Σa_n is convergent and Σb_n is divergent then $\Sigma(a_n + b_n)$ is divergent.

(iii) Sum of two divergent series need not be divergent. V.Imp

e.g. :- $\Sigma(\sin 1/n + 1/n^2) =$ divergent $= \Sigma a_n$

$\Sigma(-\sin 1/n) =$ divergent $= \Sigma b_n$

But $\Sigma(a_n + b_n) = \Sigma(\sin 1/n + 1/n^2 - \sin 1/n)$

$= \Sigma 1/n^2 \to$ convergent

Note :- (i) $\text{Max}\{a, b\} = \dfrac{a+b+|a-b|}{2}$

(ii) $\text{Min}\{a, b\} = \dfrac{a+b-|a-b|}{2}$

(iii) $\text{Max}\{x, -x\} = \dfrac{x-x+|x+x|}{2} = \dfrac{2|x|}{2} = |x|$

(iv) $\text{Min}\{x, -x\} = \dfrac{x-x-|x+x|}{2} = \dfrac{-2|x|}{2} = -|x|$

(v) $\text{Max}\{a_n, 0\} = \dfrac{a_n+0+|a_n-0|}{2} = \dfrac{a_n}{2} + \dfrac{|a_n|}{2}$

(vi) $\text{Min}\{a_n, 0\} = \dfrac{a_n+0-|a_n-0|}{2} = \dfrac{a_n}{2} - \dfrac{|a_n|}{2}$

Conditional Convergence :-

If Σa_n is a series of real numbers of mixed terms (means sign is unrestricted) we say Σa_n converges conditionally in its form if Σa_n is convergent and $\Sigma|a_n|$ is divergent.

i.e. If $\Sigma(-1)^n a_n$ is convergent but $\Sigma|a_n|$ is divergent then the series is conditionally convergent.

i.e We may have different sums of the series if we rearrange its terms.

e.g :- $\Sigma(-1)^{n-1}\frac{1}{n} = \Sigma a_n$ (say)

But $\Sigma|a_n|$ is divergent.

Note :- (i) When the series is conditionally convergent, any kind of rearrangement of terms or insert the brackets may alter the sum of the series.

(ii) A conditionally convergent series can be rearranged in a fashion that it converge to a pre-decieded numbers.

Absolute Convergence :-

If Σa_n a series of real numbers s.t. $\Sigma|a_n|$ is convergent then we say Σa_n is absolutely convergent.

Hence, Σa_n and $\Sigma|a_n|$ converges then Σa_n converges absolutely.

Note :- Absolute convergence gives liberty to insert brackets between any number of terms without changing

Result: Absolute convergence ⇒ Convergence.

i.e. If Σa_n converges absolutely then Σa_n is cgt.

i.e. S.O.P.S. of $\Sigma |a_n|$ is convergent.

i.e. $S_n = |a_1| + |a_2| + \cdots + |a_n|$

$\langle S_n \rangle$ is Cauchy

$|S_{n+p} - S_p| < \varepsilon \quad \forall\, n \in \mathbb{N}$

$$\left| \{|a_1| + |a_2| + \cdots + |a_{n+p}|\} - \{|a_1| + |a_2| + \cdots + |a_p|\} \right| < \varepsilon$$

$$\Rightarrow |a_{p+1} + a_{p+2} + \cdots + a_{p+n}| \le |a_{p+1}| + |a_{p+2}| + \cdots + |a_{p+n}| < \varepsilon$$

$$\Rightarrow |T_{n+p} - T_p| < \varepsilon$$

$\Rightarrow \langle T_n \rangle$ is Cauchy.

$\rightarrow \Sigma a_n$ is series of +ve & −ve terms.

$p_n = \max\{a_n, 0\}$

$q_n = \min\{a_n, 0\}$

If Σa_n is conditionally convergent, Σp_n & Σq_n are divergent.

$$p_n = \frac{a_n + 0 + |a_n - 0|}{2} = \frac{a_n + |a_n|}{2}$$

$$q_n = \frac{a_n + 0 - |a_n - 0|}{2} = \frac{a_n - |a_n|}{2}$$

Since $\Sigma |a_n|$ is divergent

$\Rightarrow \Sigma \left[\frac{a_n + |a_n|}{2}\right] = \Sigma p_n$ & $\frac{1}{2}\left[\frac{a_n - |a_n|}{2}\right] = \Sigma q_n$ are divergent.

Note:- (i) If $\sum a_n$ is absolutely convergent then subseries of +ve terms and -ve terms are convergent.

(ii) If $\sum a_n$ is conditionally convergent then subseries of +ve terms and -ve terms are divergent.

E.g.:- $\sum \sin \frac{1}{n^2}$ is absolutely convergent.

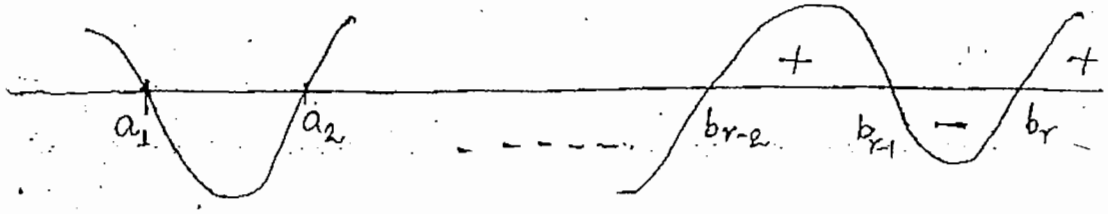
Wavy Curve Method For Sign Convention:-

$$f(x) = y = \frac{(x-a_1)(x-a_2)\cdots(x-a_k)}{(x-b_1)(x-b_2)\cdots(x-b_s)}$$

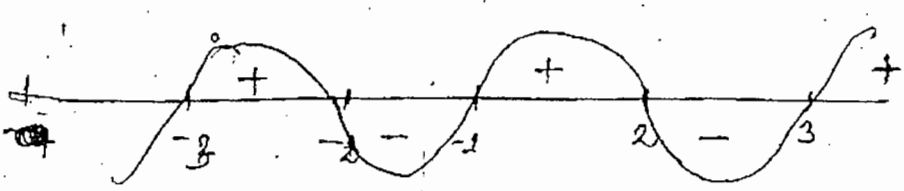
where $a_1 < a_2 < \cdots < a_k < b_1 < b_2 < \cdots < b_s$

$$\text{i.e } y = \frac{\pi(x-a_k)}{\pi(x-b_s)}$$

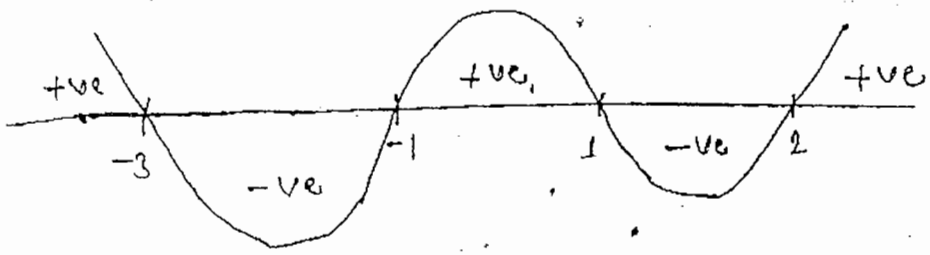
called wavy curve method. Draw the points a_k and b_r on the real line in their natural order and draw the wavy curve on the real line and give the sign +, −, +, −, alternatively starting from right.



E.g.:- $f(x) = \frac{(x-1)(x-2)(x+2)}{(x-3)(x+3)}$



Eg :- $f(x) = \sqrt{\dfrac{(x-1)(x+1)}{(x-2)(x+3)}}$ → cannot be −ve for domain.

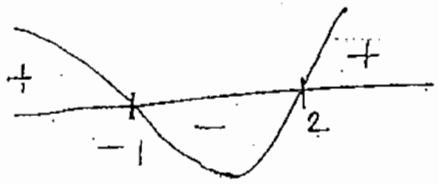


Domain = $(-\infty, -3) \cup [-1, 1] \cup (2, \infty)$

OR, $\mathbb{R} - [-3, -1) \cup (1, 2]$

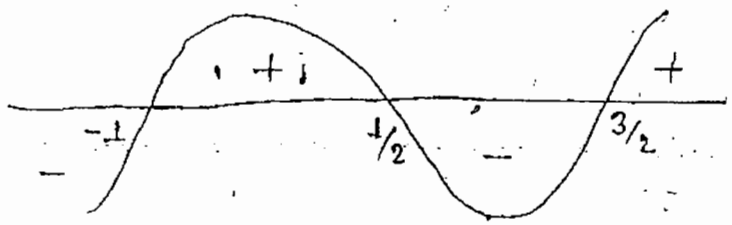
Note :- Even powers do not effect the sign.

Eg :- $f(x) = \dfrac{(x-1)^{2} (x-2)^{3}}{(x+1)(x-3)^{4}}$



Eg :- $f(x) = \sqrt{(2x-3)(x+1)(1-2x)}$

$= \sqrt{-4(x-3/2)(x+1)(x-1/2)}$



Domain = $(-\infty, -1] \cup [1/2, 3/2]$

Or, $\mathbb{R} - (-1, 1/2) \cup (3/2, \infty)$

→ $\log_{10} \frac{(x-1)(x-2)(x+2)}{(x-3)(x+3)}$

Domain $(-3,-2) \cup (1,2) \cup (3,\infty)$

→ **Domain**

(i) $y = \log x$ $(0,\infty)$

(ii) $y = \sin^{-1}x$ $[-1,1]$

(iii)* $y = \sqrt{(x-1)(x-2)}$ $(-\infty,1) \cup [2,\infty)$

(iv) The domain of $\sin^{-1}x$ and $\cos^{-1}x$ are same but range of $\sin^{-1}x$ is $[-\pi/2, \pi/2]$ and range of $\cos^{-1}x$ is $[0,\pi]$.

(v) $y = \log(x-1)(x-2)$ $(-\infty,1) \cup (2,\infty)$.

Domain:-

1). $\sqrt{f(x)}$; $f(x) \geq 0$

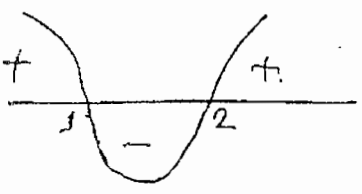
2). $\log_{g(x)} f(x)$; $f(x) > 0$
$g(x) > 0$ & $g(x) \neq 1$

3). Different t-ratios / inverse t-ratios
$\sin x$, $\cos x$, $\tan x$, $\sin^{-1}x$, $\cos^{-1}x$ etc.

e.g:- $y = \log_{2x-1}(x-2)(1-x)$

Here $f(x) = (x-2)(1-x)$

$= -(x-2)(x-1) > 0$



$g(x) = 2x-1 > 0$ if $x > 1/2$ and $g(x) \neq 1$.

$\Rightarrow 2x-1 \neq 1$

$\Rightarrow \boxed{x \neq 1}$

$x \in (1/2, \infty) - \{1\}$

Domain $= ((1/2, \infty) - \{1\}) \cap (1, 2)$

$= (1, 2)$

e.g.: $y = \log_{10} \underbrace{\log_{10} \log_{10} \log_{10}}\, x$

$\Rightarrow \log_{10} \log_{10} \log_{10} x > 0$

$\Rightarrow \log_{10} \log_{10} x > 1$

$\Rightarrow \log_{10} x > 10$

$\Rightarrow x > 10^{10}$

Domain $= (10^{10}, \infty)$ (Ans).

Note:-(i) $y = \underbrace{\log_a \log_a \log_a \cdots \log_a}_{m\text{-times}} x$

Then Domain $= \left(a^{a^{a^{\cdots^{a}}}}, \infty\right)$ (m-2 times)

(ii) $\log_a x > 0$; $a > 1$, $x > 1$

(iii) $\log_a x < 0$; $a > 1$, $x < 1$

(iv) $\log_a x = 0$; $a > 1$, $x = 1$.

(v) $a > 1$, $\log_a x > 0$; $x > 1$

(vi) $a < 1$, $\log_a x < 0$; $x > 1$

(viii) $a<1$, $x<1$, $\log_a x > 0$.

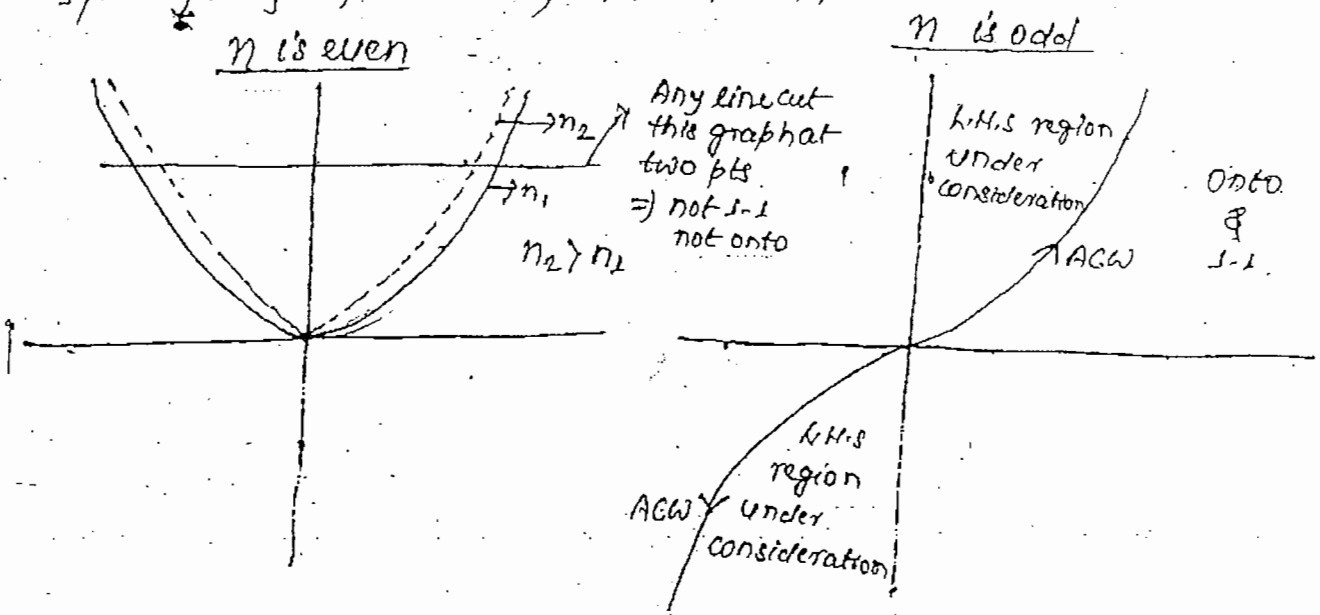
GRAPHS

Def: Let $f: S \to \mathbb{R}$, where $S \subseteq \mathbb{R}$ then $(x, f(x))$ is called graph of $y=f(x)$.

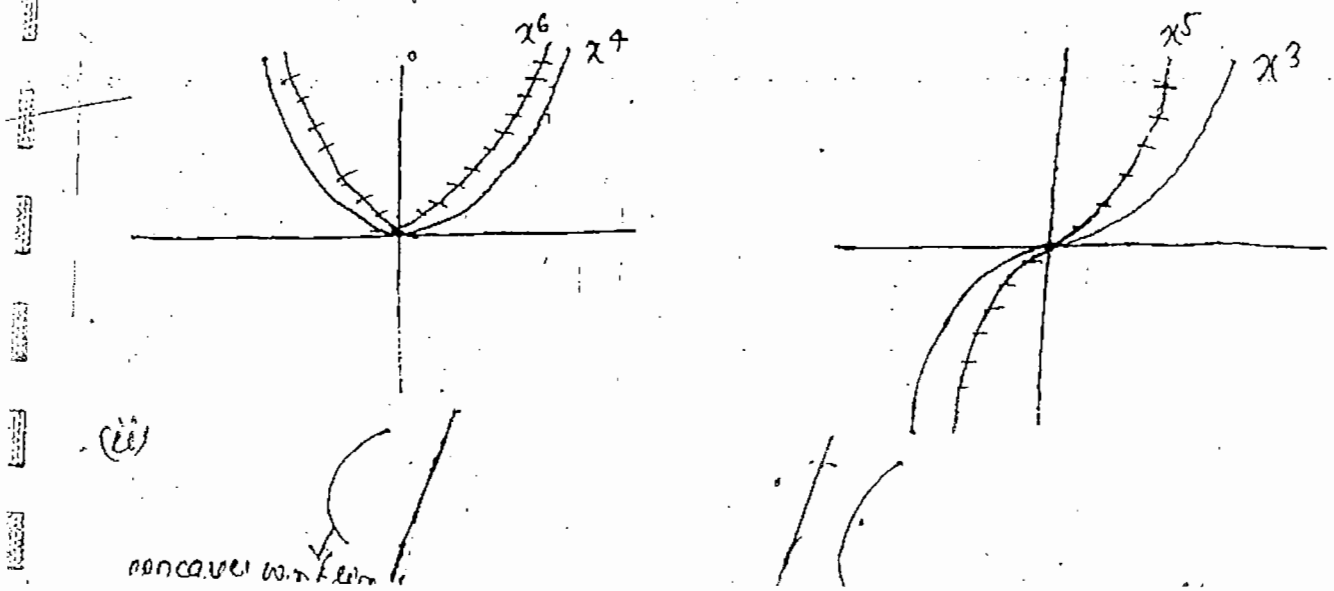
Elements of domain are called arguments.

Some Important Graphs :-

1) $y = f(x) = x^n$; $n \in \mathbb{N}$, $n > 1$

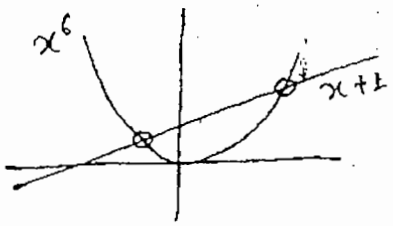


Note: (i) The branches of graph get closer to y-axis as n - increasing



(ii)

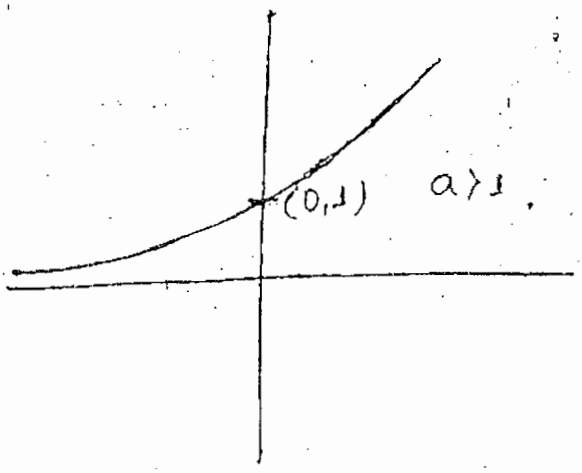
Q.7: $x^6 - x - 1$ (Gate)

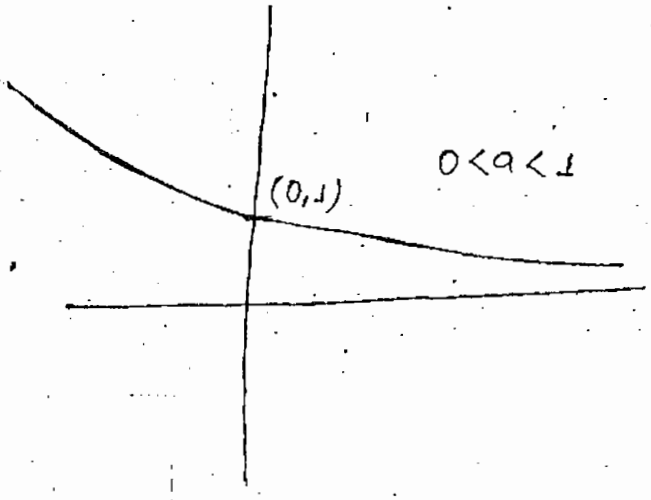


$x^6 = x+1$

2). $y = f(x) = a^x$; $a > 0$ and $a \neq 1$ i.e. $a \in (0,\infty) - \{1\}$

Exponential

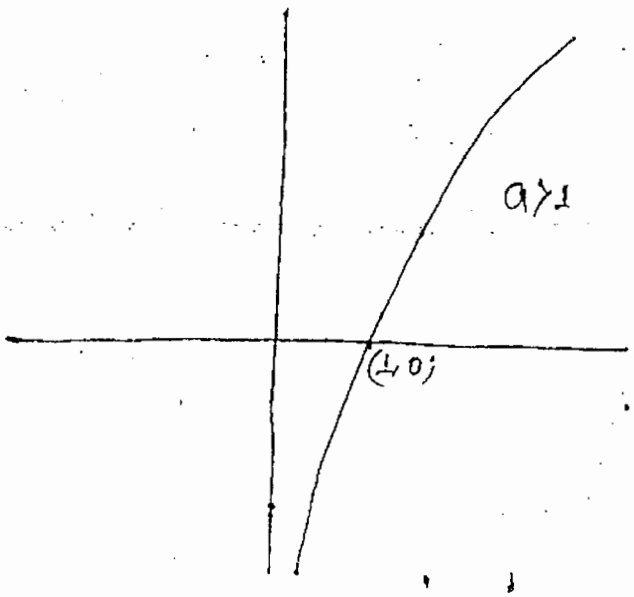


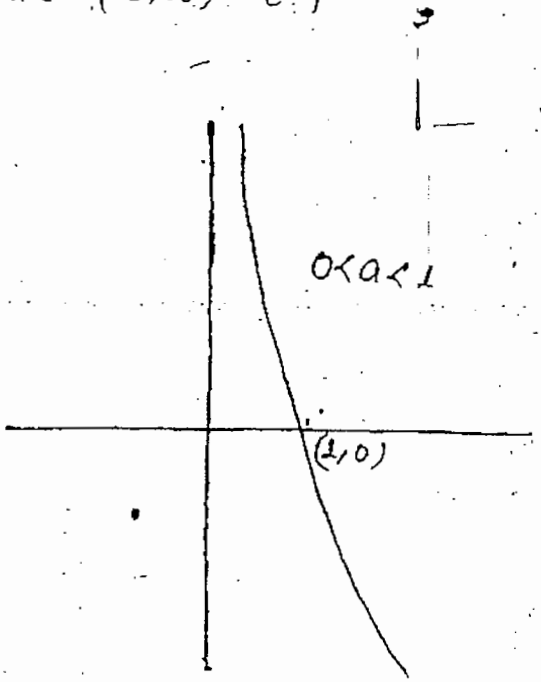


3): $y = f(x) = \log_a x$; $a > 0$, $a \neq 1$, $x > 0$

i.e $x \in (0,\infty)$

$a \in (0,\infty) - \{1\}$





Rules:-

(i) If a line parallel to x-axis cuts the graph more than once, function is not one-one i.e function is many-one.

(ii) If there exists a line which does not cut the curve at all and passing through the co-domain section, then the function is not onto.

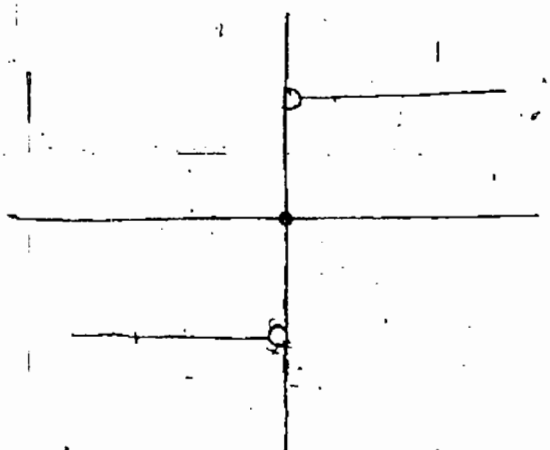
x-axis = domain section.

y-axis = co-domain section

→ If $f: \mathbb{R}^+ \to \mathbb{R}^+$ then f is one-one and onto.

4) $sig(x) = \frac{|x|}{x} = \begin{cases} -1, & x<0 \\ 0, & x=0 \\ 1, & x>0 \end{cases}$ Step function

$sig(f(x)) = \frac{|f(x)|}{f(x)} = \begin{cases} -1, & f(x)<0 \\ 0, & f(x)=0 \\ 1, & f(x)>0 \end{cases}$



Remark: (i) If function is decreasing then inverse of that function is also decreasing.

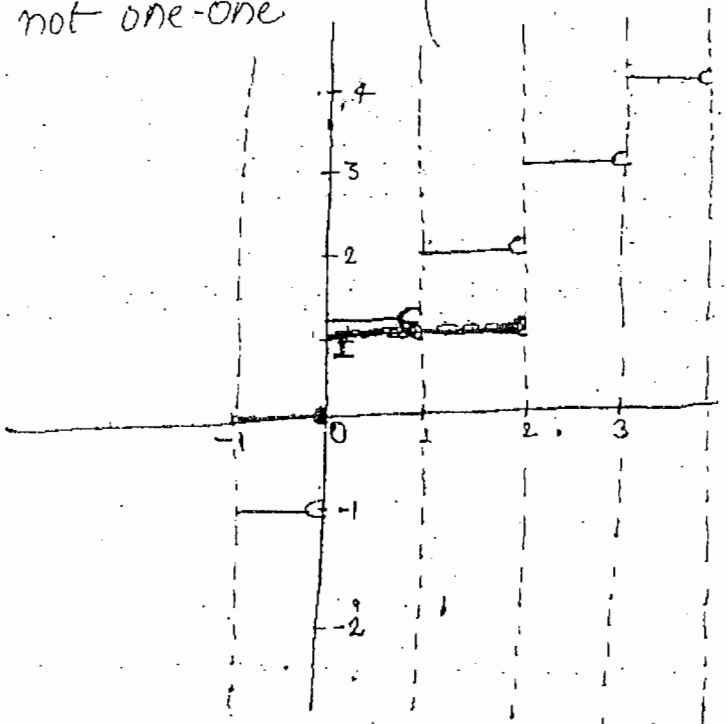
eg:- $\log x$, e^x etc.

(ii) Periodic ⇒ not algebraic
& Algebraic ⇒ not periodic.

Eg: $\{x\} = x - [x]$ → not algebraic ⇒ transcedental.

5) $f(x) = [x] = \begin{cases} -1 & ; -1 \le x < 0 \\ 0 & ; 0 \le x < 1 \\ 1 & ; 1 \le x < 2 \\ n & ; n \le x < n+1 \end{cases}$ — Range = Integers only.

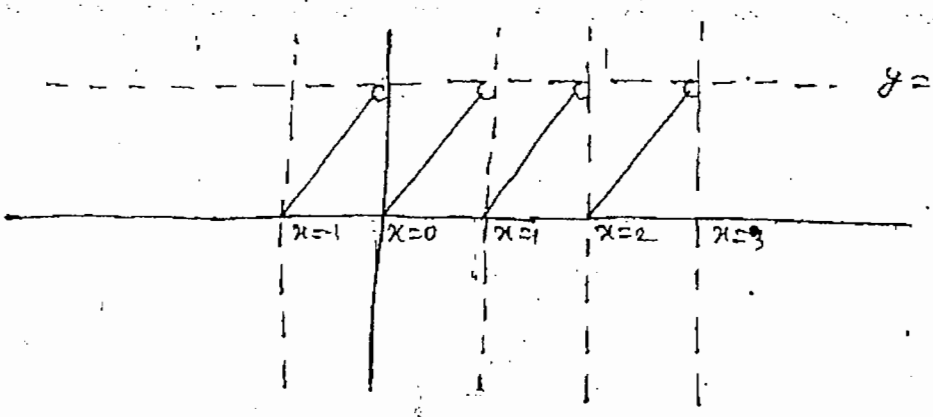
Step function
not periodic
monotonic but
not one-one



6) $y = f(x) = \{x\} = x - [x]$ = $\begin{cases} x+1 & ; -1 \le x < 0 \\ x & ; 0 \le x < 1 \\ x-1 & ; 1 \le x < 2 \\ x-2 & ; 2 \le x < 3 \end{cases}$

period = 1

periodic

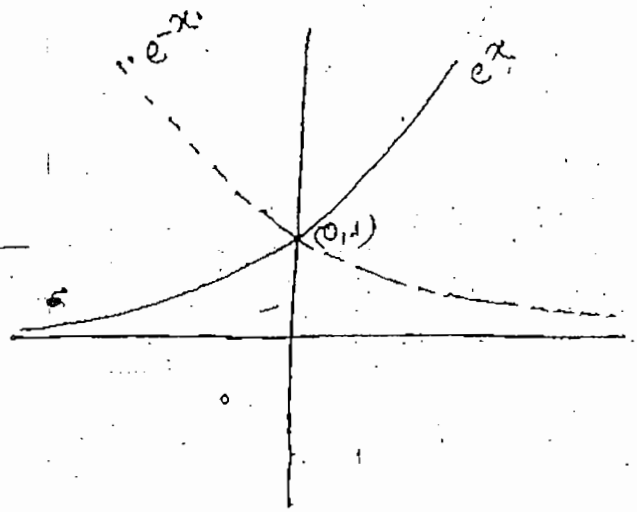


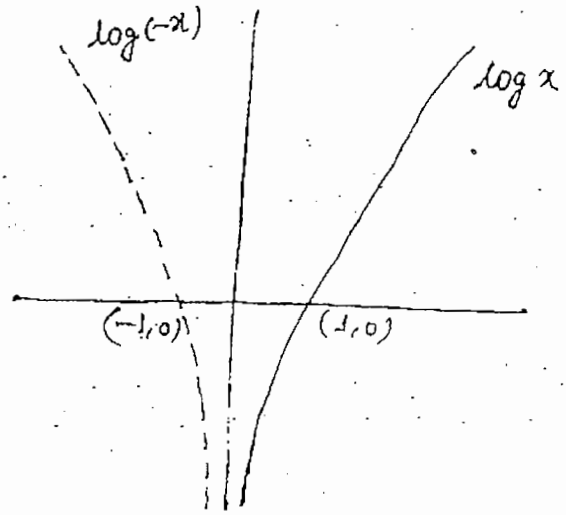
many-one

#). $y = f(x)$,
$y = f(-x)$

Take the reflection of the graph of $y = f(x)$ in the line y-axis. The image is the desired curve.

e.g.:

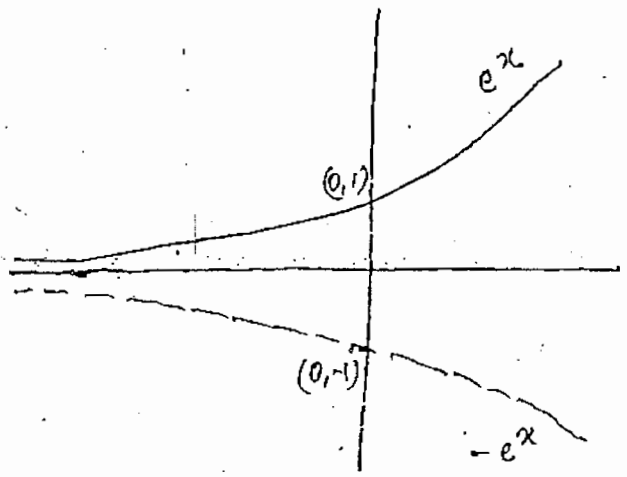


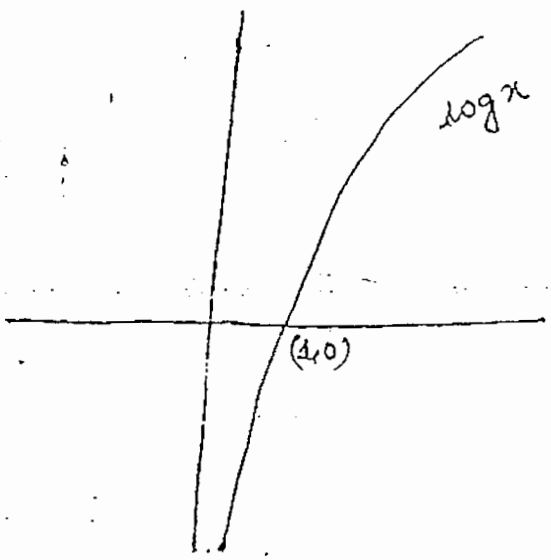


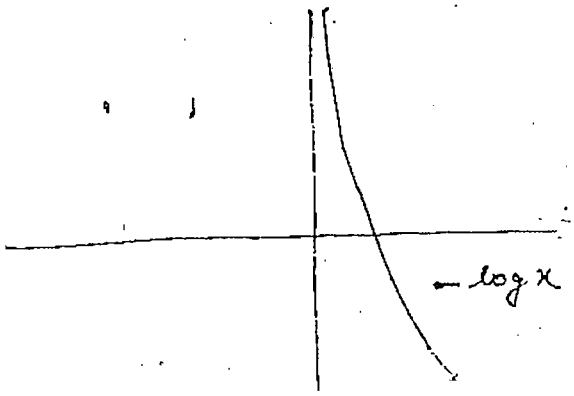
8). $y = f(x)$, $y = -f(x)$

Draw $y = f(x)$ and take the reflection into x-axis.

e.g.:-

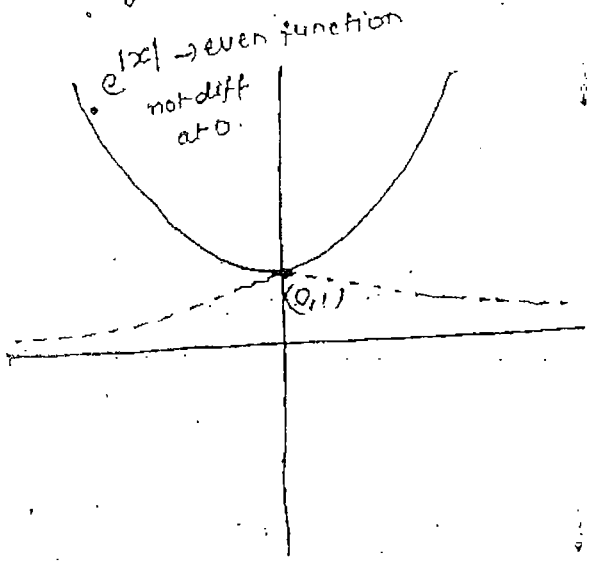


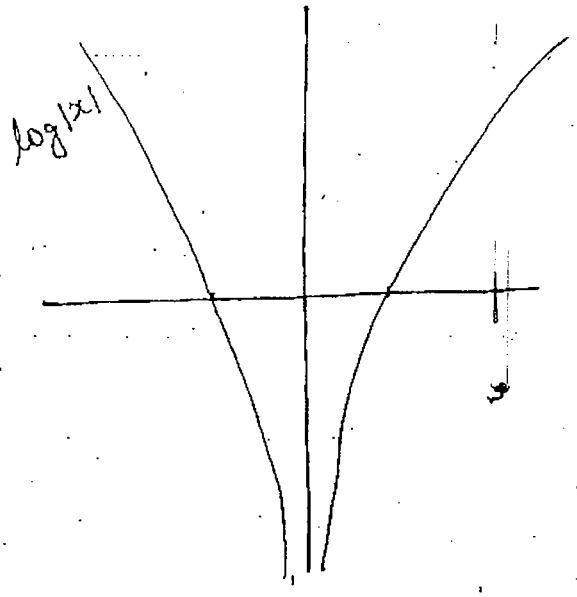


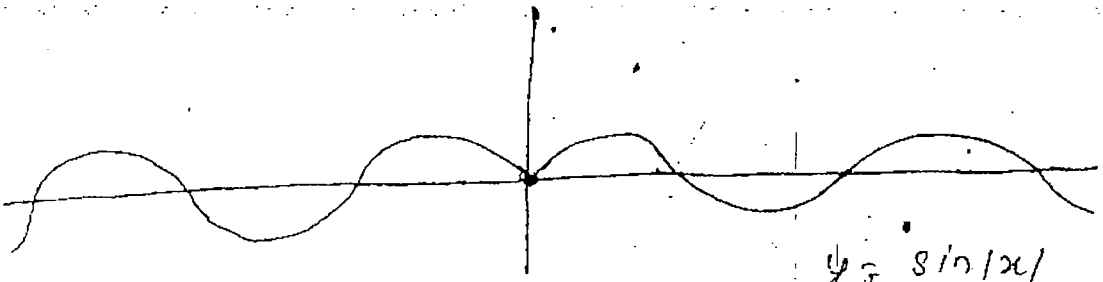


9). $y = f|x|$

Draw $y = f(x)$ only for +ve values of x i.e. only right side of y-axis, and take the reflection of this curve into y-axis. Object and image together, represents the desired curve.





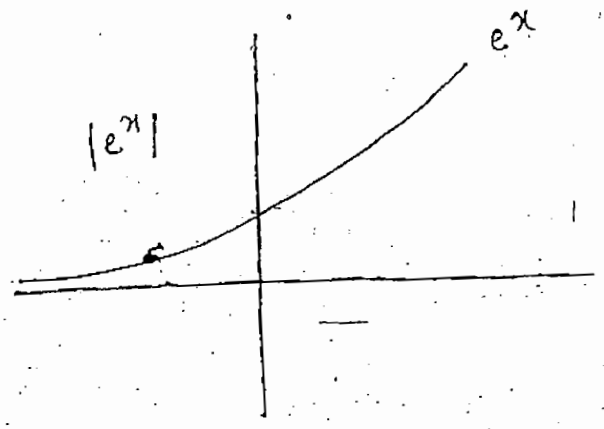


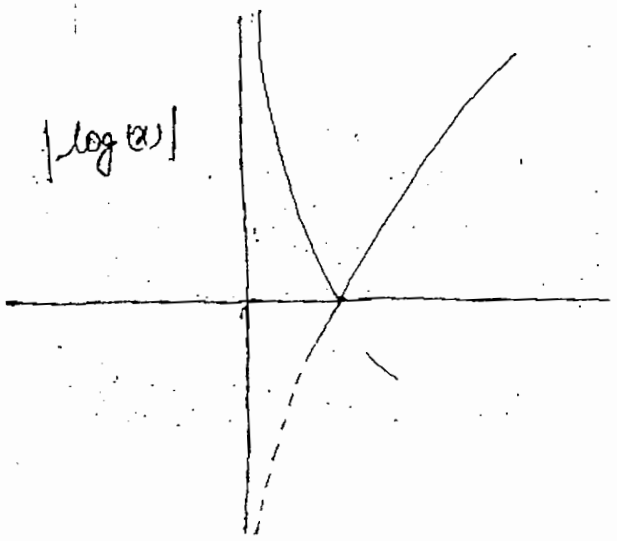
$y = \sin|x|$

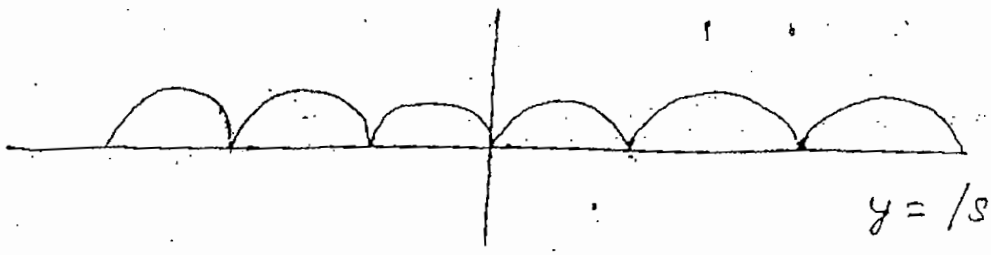
not differentiable at 0.

10) $y = |f(x)|$

Draw $y = f(x)$ and take the reflection of the portion line below x-axis into x-axis and delete the object image and retained part together represents the desired curve.

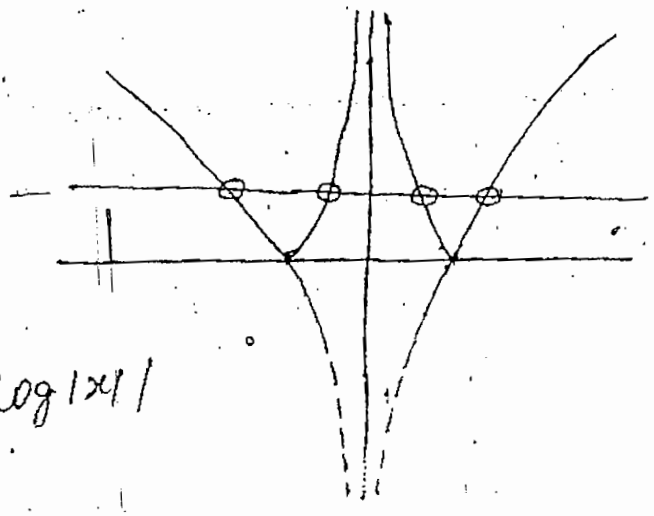






$y = |\sin|x||$

e.g:- $|\log|x|| = 1$. then no. of roots = ?



Take $y_1 = |\log|x||$

and $y_2 = 1$

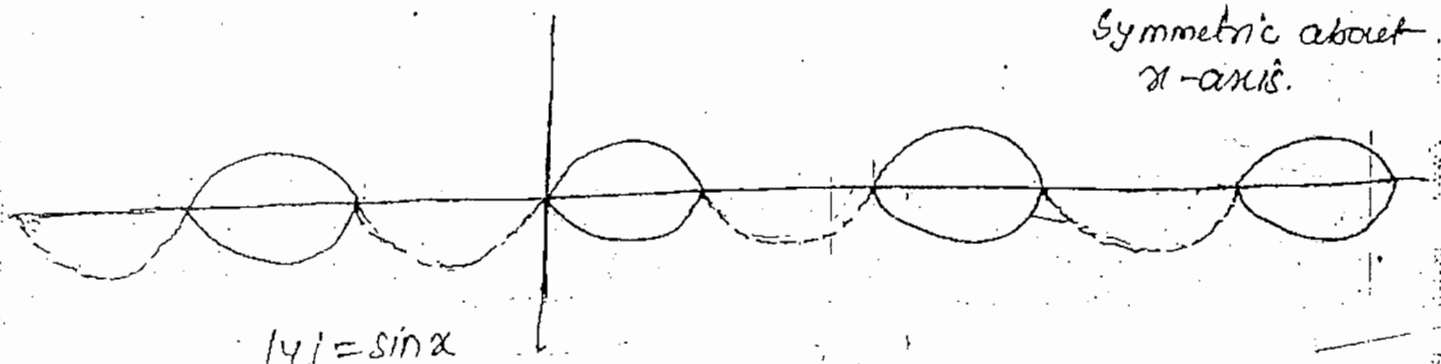
No. of pts of intersection = No. of roots

Hence no. of roots = 4

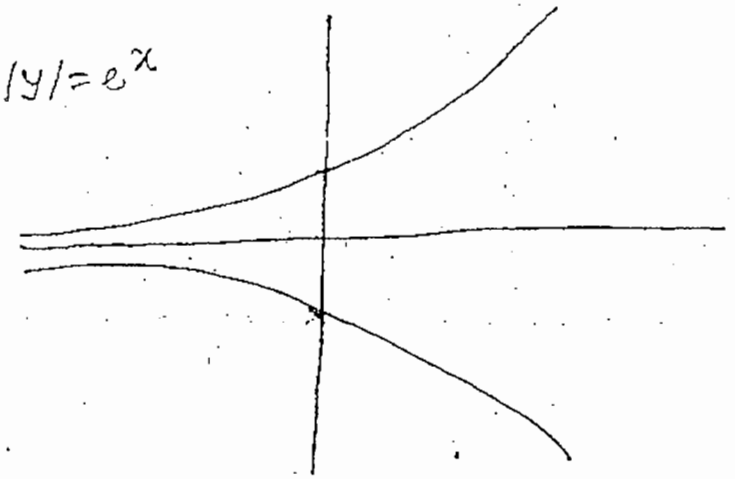
11) $|y| = f(x)$ ⟨ $-y = f(x)$ / $y = f(x)$

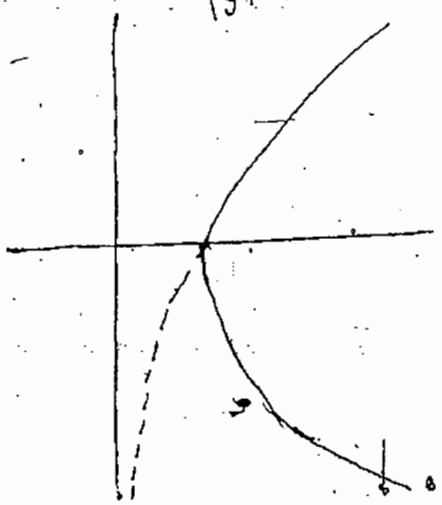
It is not a function.
It is only relation

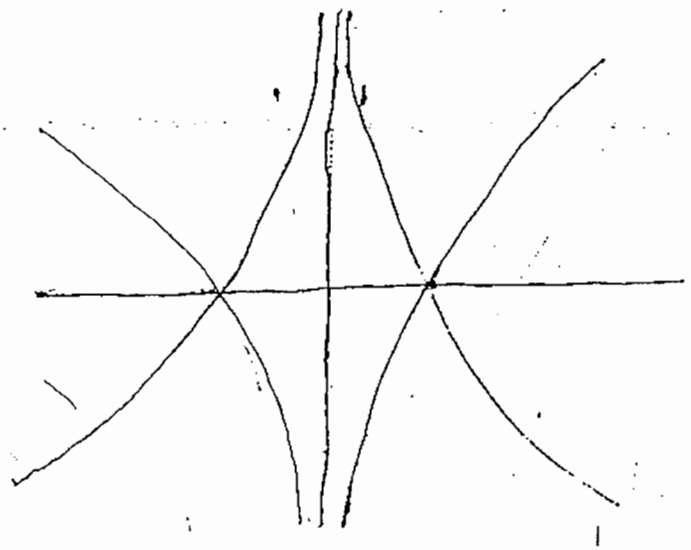
Draw $y = f(x)$ and take the reflection of the portion line above x-axis into x-axis. Object and image together represents the desired curve.



$|y| = \sin x$



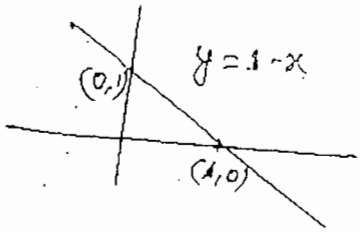




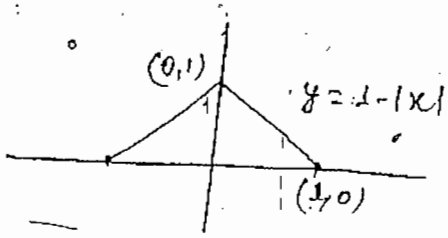
$|y| = |\log|x||$

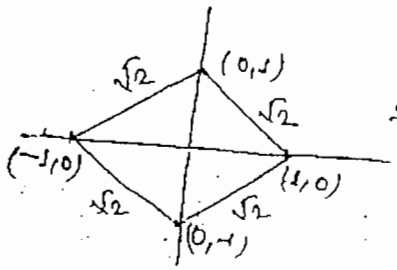
→ $|x|+|y|=1=a$

We take $y = 1-x = f(x)$



$y = 1-|x|$





Square

$|x|+|y|=1$.

Note:- (i) $|x|+|y|=a$ is a square of area $= 2a^2$

(ii) $|x-\alpha|+|y-\beta|=a$ is a square of area $= 2a^2$ and side is $|a|$

e.g.:- (i) $|x-1|+|y-3|=5$, Here $a=5$

Area of the square $= 2a^2 = 2\times5^2 = 2\times25 = 50$

(ii) $|x|+|y|=1$, Here $a=1$.

Area of the square $= 2a^2 = 2\times1^2 = 2$

(iii) The circle $(x-1)^2+(y-1)^2=25$ has centre $(1,1)$ and radius 5.

Area of circle $= 25\pi$

The circle $x^2+y^2=25$ has area $\underline{25\pi}$.

Note:- $\ln x = \log_e x$

$\log x$ means $\log x$ to the base 10 or any no. to the base is given.

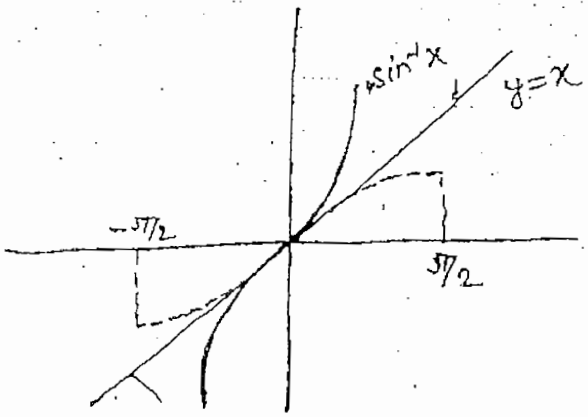
12) $y = f^{-1}(x)$

Draw the graph of $y = f(x)$ is that segment of domain where f is bijective and take the reflection of this curve in the line $y = x$. Image is the desired curve.

→ $f: [-\pi/2, \pi/2] \to [-1, 1]$

$f(x) = \sin x \Rightarrow f^{-1}(x) = \sin^{-1} x$

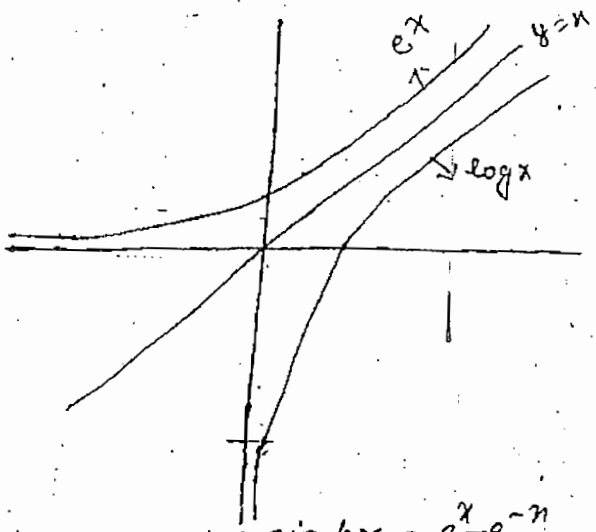
f is 1-1, onto and invertible



→ $f: \mathbb{R} \to \mathbb{R}^+$

$f(x) = e^x \Rightarrow f^{-1}(x) = \log x$

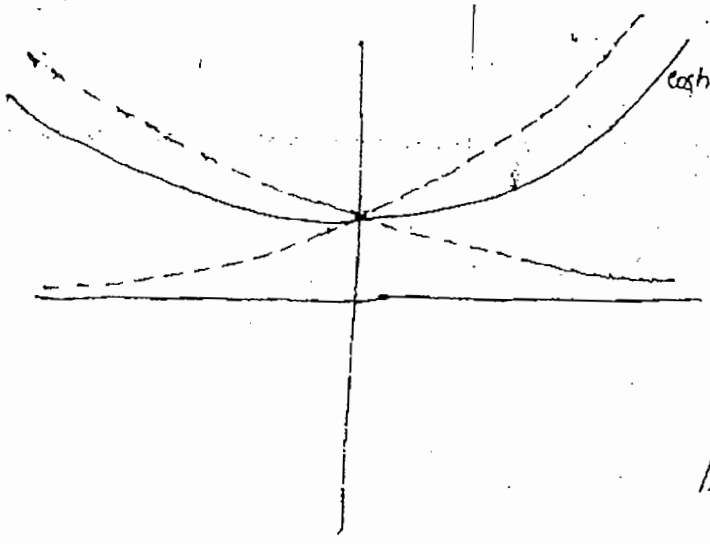
f is 1-1, onto and invertible



→ $\cosh x = \frac{e^x + e^{-x}}{2} = A.M.\ (e^x, e^{-x})$

Neither 1-1 nor onto

not invertible

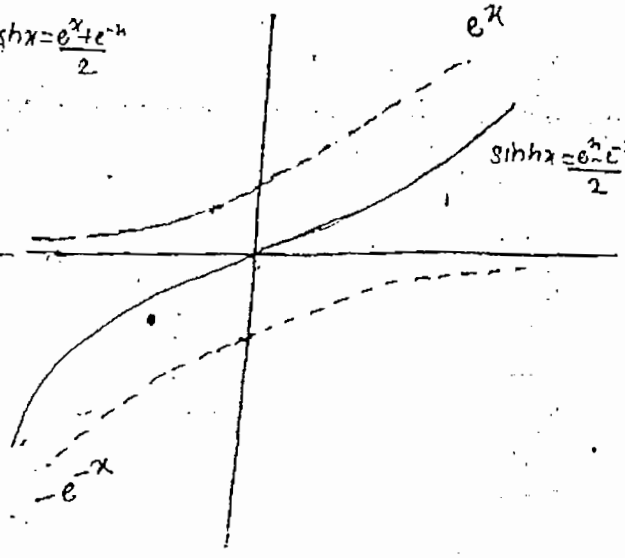


→ $\sinh x = \frac{e^x - e^{-x}}{2}$

$A.M.$ of $(e^x, -e^{-x})$

one-one & onto

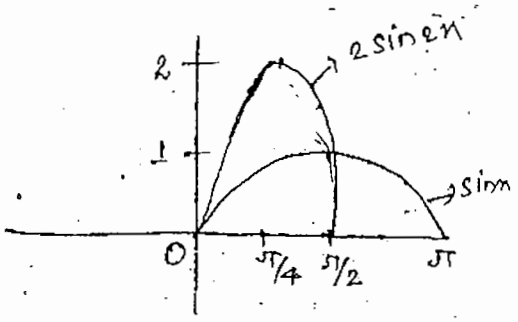
$\Rightarrow$ invertible.



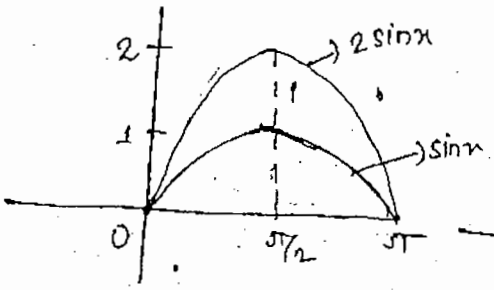
13). (i) $y = af(x)$ (ii) $y = f(ax)$

(iii) $y = f(x) + a$ (iv) $y = f(x+a)$

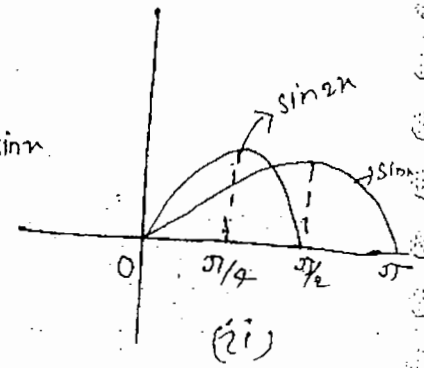
(v) $y = af(ax)$



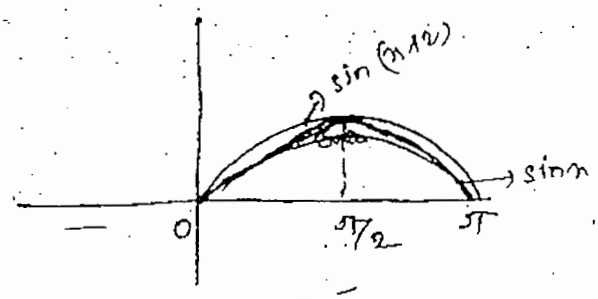
(v)



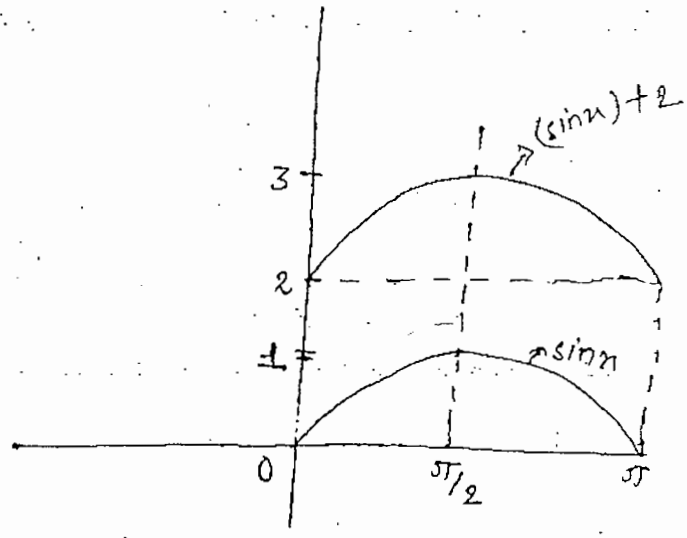
(i)



(ii)



(iv)



(iii)

Problem:- $\sin x = a + \frac{1}{a}$, then no. of solutions ?

Solution:- We know that

$$G.M \leq A.M$$

$$\sqrt{a \cdot \frac{1}{a}} \leq \frac{a + \frac{1}{a}}{2} \quad ; a > 0$$

$$\Rightarrow 1 \cdot 2 \leq a + \frac{1}{a} \quad \Rightarrow 2 \leq a + \frac{1}{a}$$

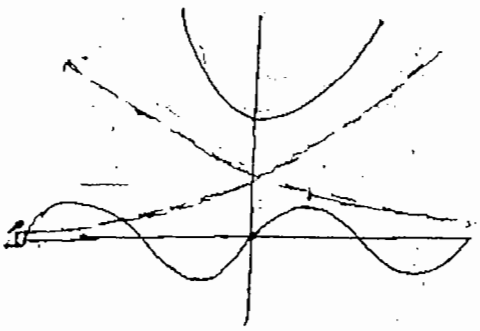
But $-1 \leq \sin x \leq 1$

Then no solution exists

Problem:- $\sin x = e^x + e^{-x}$

Solution:- $\sqrt{e^x \cdot e^{-x}} \leq \frac{e^x + e^{-x}}{2}$

$\Rightarrow 2 \leq e^x + e^{-x}$



But $-1 \leq \sin x \leq 1$

Then no solution exists.

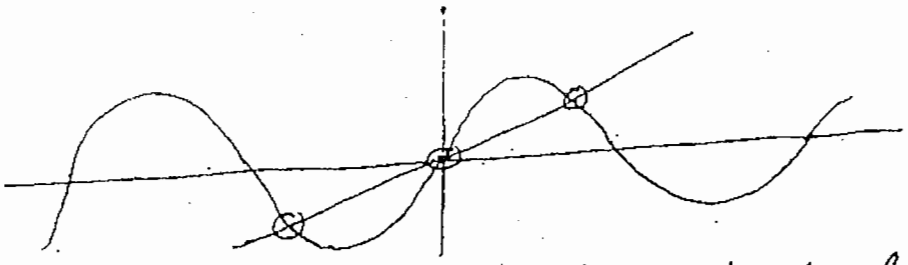
Problem:- Show that $3\sin x = x$ has finite number of roots. or solutions.

$\Rightarrow \sin x = x/3$ — (A)

Solution:- Let $y_1 = \sin x$ — (1)

$y_2 = x/3$ — (2)

x	y
1	$1/3$
2	$2/3$
0	0
-1	$-1/3$



Since graph of (1) and (2) cut at finite number of points

$\Rightarrow$ (A) has finite number of roots.

i.e. (A) has 3 roots.

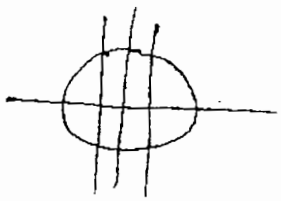
Note some points on graph:-

Note 1:- If the line $x=a$ does not intersect the curve $y=f(x)$ from both sides i.e. there is break on the curve at $x=a$ then we say the point $x=a$ is point of discontinuity.

Note 2:- If there is a line parallel to y-axis intersect the curve more than one, then the curve represent a relation but not a function.

e.g:- $x^2+y^2=1$.

Relation but not function



Note 3:- If a line // to x-axis passing through the co-domain (section of y-axis) intersect the curve more than one then function fails to be one-one; i.e function is many-one.

Note 4:- If the same line as above does not intersect the curve, then function fails to be onto.

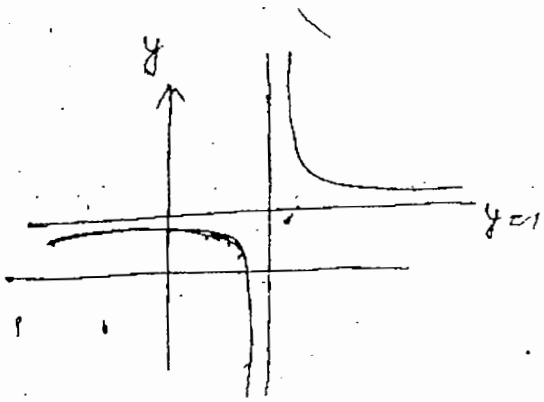
e.g:- $f: \mathbb{R}-\{1\} \to \mathbb{R}$

$$y = \frac{x+1}{x-1} = \frac{x-1+2}{x-1}$$

$$\Rightarrow y = 1 + \frac{2}{x-1}$$

$$\Rightarrow y-1 = \frac{2}{x-1}$$

$$\Rightarrow (y-1)(x-1) = 2$$



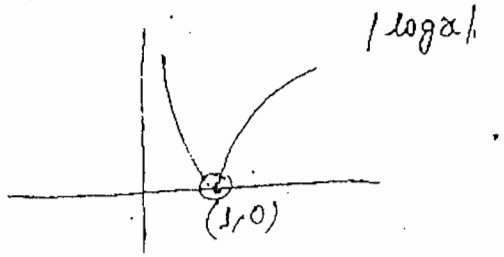
} i.e f is one-one but not onto.

→ If $f: \mathbb{R} \to \mathbb{R}-\{1\}$

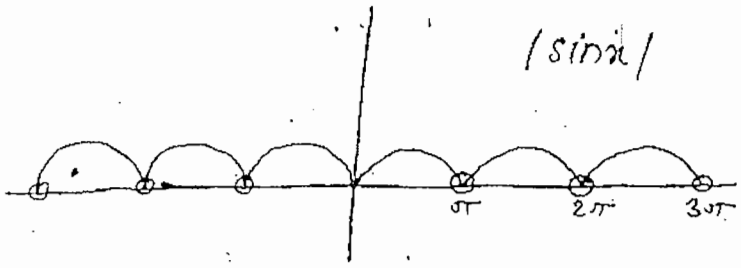
then f is one-one & onto ⇒ f is bijective.

Note 5:- If at $x=a$, there is a sharp edge on the curve then function fails to be differentiable at $x=a$.

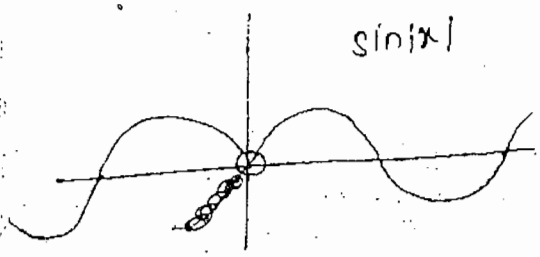
eg:



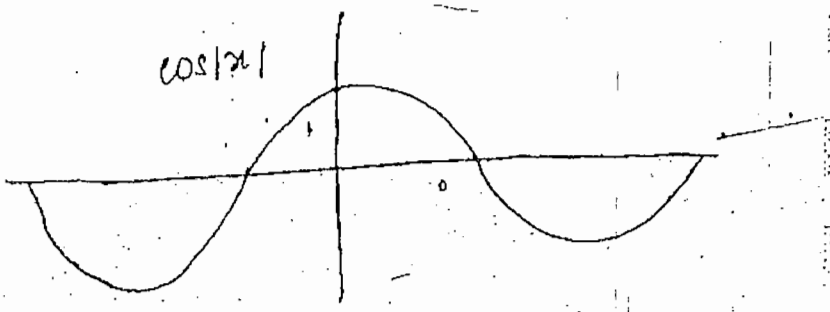
Not differentiable at $x=1$.



Not diff at $x = n\pi$, $n \in \mathbb{Z}$.



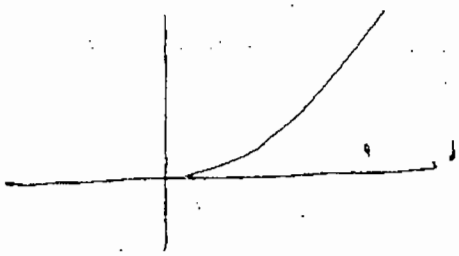
Not diff' at $x=0$



Differentiable everywhere

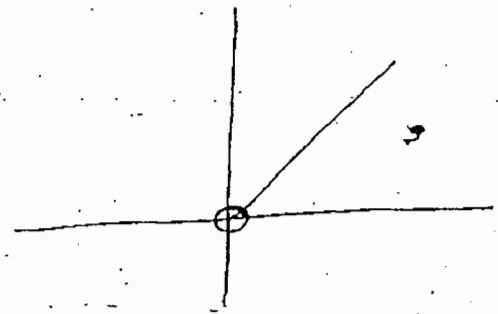
eg:- (i) $f(x) = x(x+|x|)$

$$= \begin{cases} 0 & ; x \le 0 \\ 2x^2 & ; x \ge 0 \end{cases}$$



diff at $x=0$

(ii) $f(x) = \begin{cases} 0 & x \ge 0 \\ 4x & x \le 0 \end{cases}$



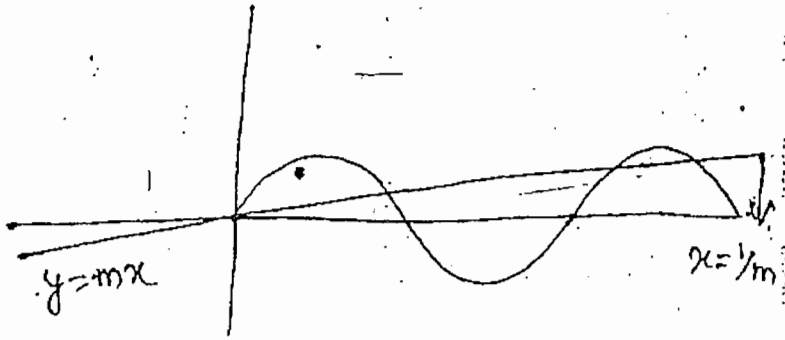
Not diff. at $x=0$

Note 57:-

$y = mx$

$y = \sin x$

$\sin x = mx$

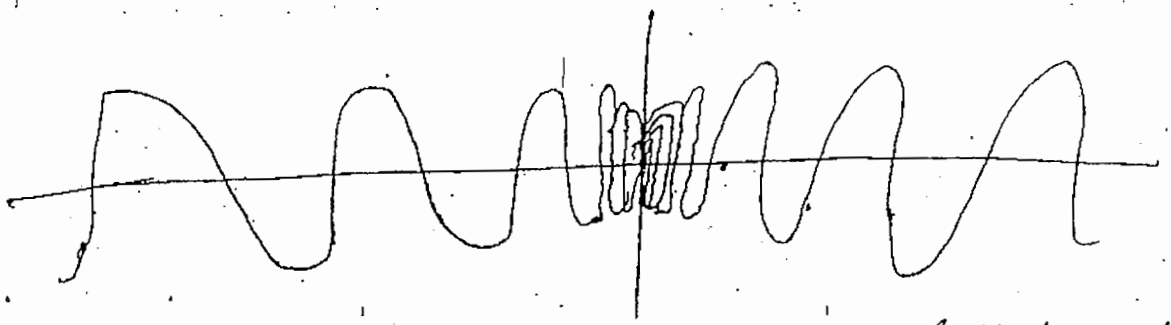


→ 7 intersection points i.e. 7 roots (in case $m = 1/10$)
$\sin x = mx$ will always have finite number of roots when $m \neq 0$.

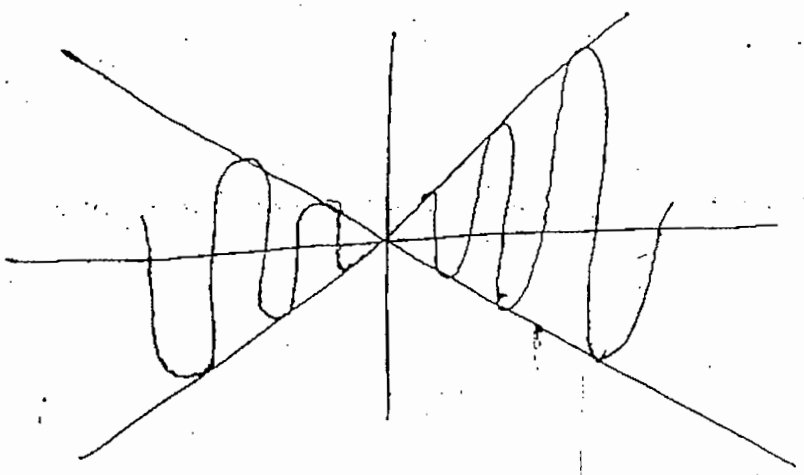
→ If $m = 1$ or $m > 1$ then only one root.

Note 6:- When x - approaches to 'a' and the graph tends to a line // to y-axis then the function fails to be <u>uniformly continuous</u> in every that interval whose either of end point is 'a'.

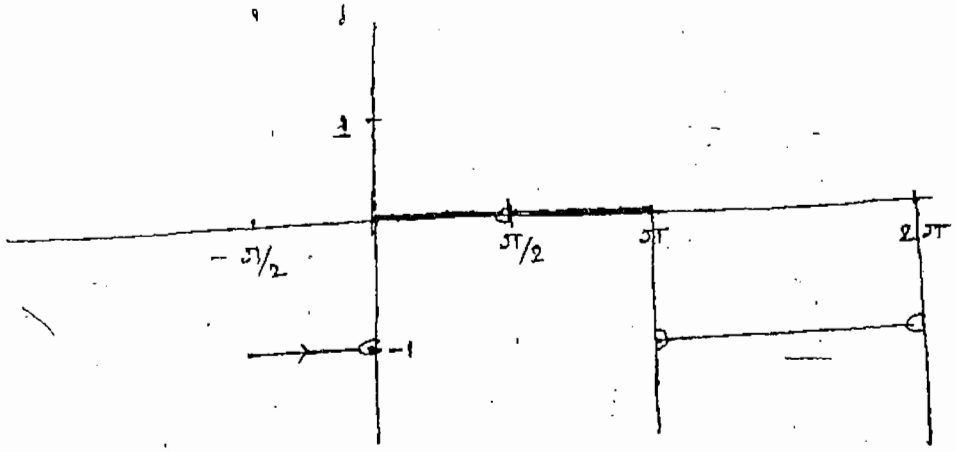
e.g.:- $f(x) = \sin\left(\frac{1}{x}\right)$ is not uniformly continuous in $(0, 1)$.



e.g.: $\sin\left(\frac{1}{x}\right) = x$ in $(0,1)$ countably infinite roots.



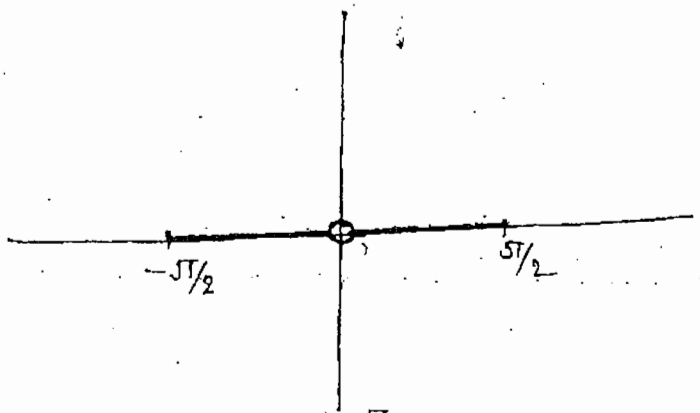
Example :- ① $y = [\sin x]$



At 0

L.H.S $\lim \to -1$, $\lim \to 0$ $\Rightarrow$ Limit does not exist at 0.

Example :- ② $y = [\cos x]$



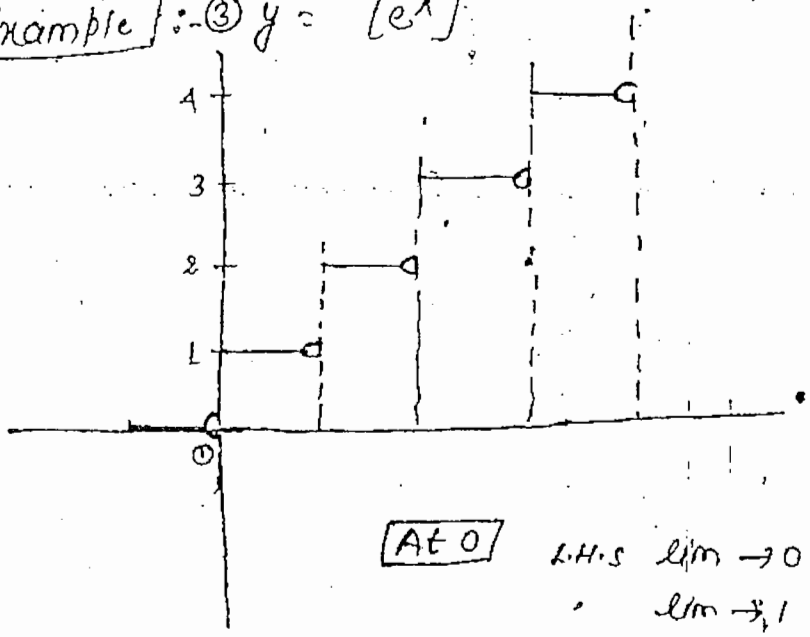
At 0

L.H.S $\Rightarrow \lim \to 0$

R.H.S $\Rightarrow \lim \to 0$

$\Rightarrow$ Limit exists at 0

Example :- ③ $y = [e^x]$



$y = e^x$

for $y = 1 \Rightarrow x = 0$

for $y = 2 \Rightarrow x = \log 2$

for $y = 3 \Rightarrow x = \log 3$

⋮

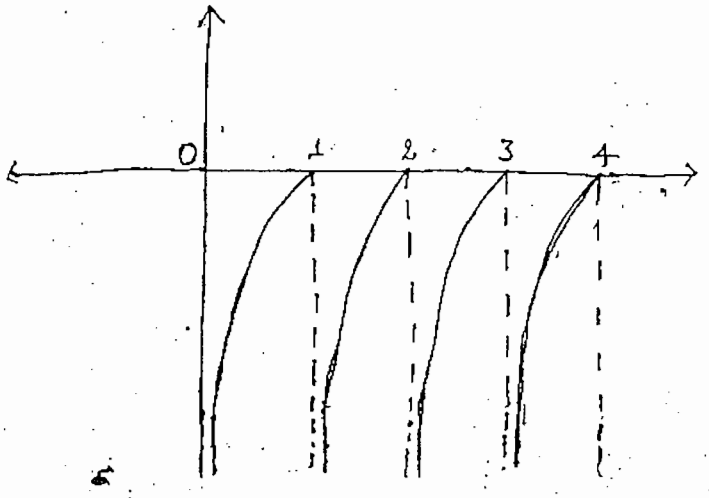
for $y = n \Rightarrow x = \log n$

$\log_e 2 = 0.6931$

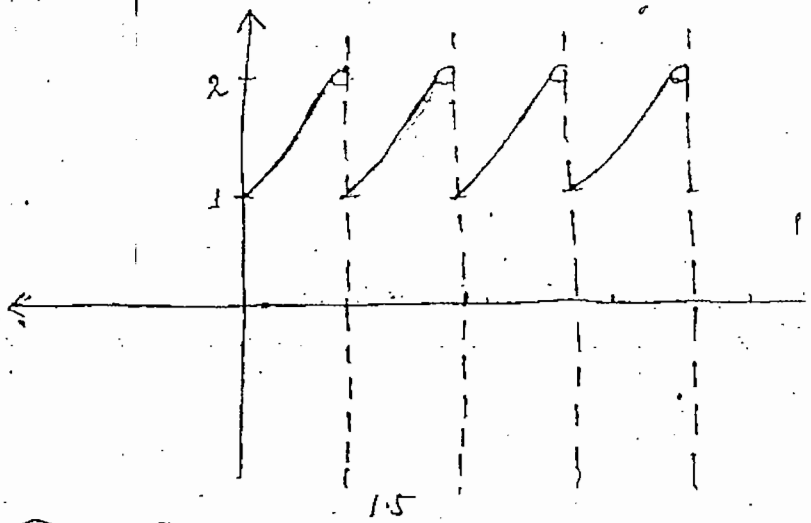
$\log_e 3 = 1.0986$

At 0 L.H.S $\lim \to 0$, $\lim \to 1$ $\Rightarrow$ 0 is not l.p of $y = [e^x]$

Example :- ④ $y = \log(x - [x]) = \log\{x\}$



Example :- ⑤ $y = 2^{x-[x]} = 2^{\{x\}} = \dfrac{2^x}{2^{[x]}}$



Example : ⑥ $\int_0^{1.5} [x^2]\, dx$

Solution :- $\int_0^{1.5} [x^2]\, dx$

$$= \int_0^1 0\, dx + \int_1^{\sqrt{2}} 1\, dx + \int_{\sqrt{2}}^{1.5} 2\, dx$$

$$= 0 + (x)_1^{\sqrt{2}} + 2(x)_{\sqrt{2}}^{1.5}$$

$$= \sqrt{2} - 1 + [illegible] + 2(1.5 - \sqrt{2}) = \sqrt{2} - 1 + 3 - 2\sqrt{2}$$

$$= 2 - \sqrt{2} \quad \text{Ans}$$

Example: (7) $\int_0^2 [e^x]\,dx$

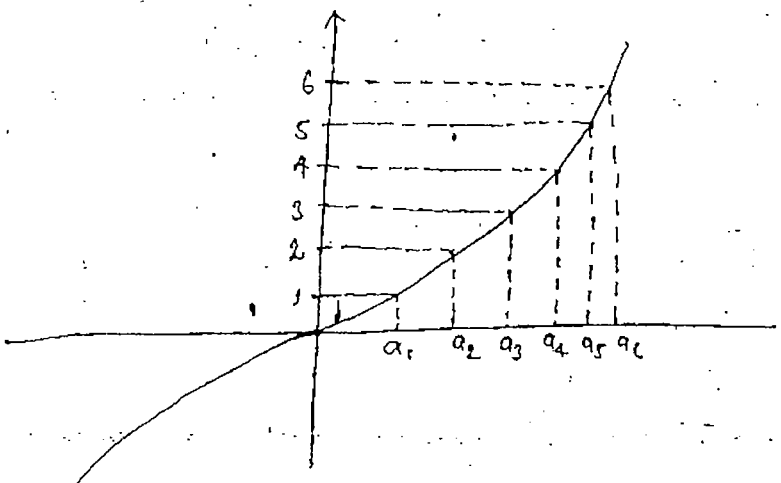
Solution: $\int_0^2 [e^x]\,dx$

$$= \int_0^{\log 2} 1\,dx + \int_{\log 2}^{\log 3} 2\,dx + \int_{\log 3}^{\log 4} 3\,dx + \int_{\log 4}^{\log 5} 4\,dx + \int_{\log 5}^{\log 6} 5\,dx + \int_{\log 6}^{\log 7} 6\,dx + \int_{\log 7}^{2} 7\,dx$$

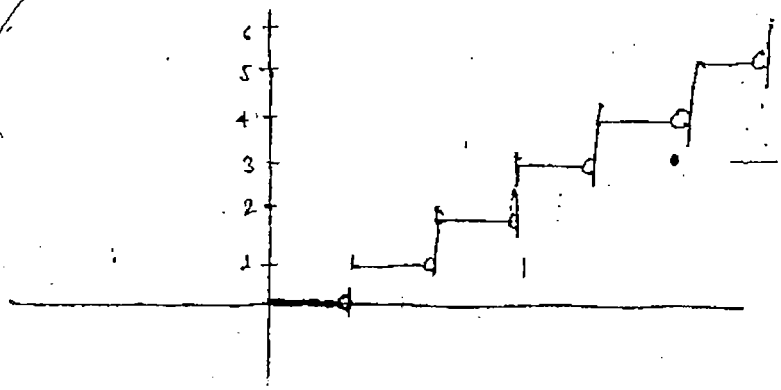
$$= \log 2 + 2\log 3 - 2\log 2 + 3\log 4 - 3\log 3 + 4\log 5 - 4\log 4 + 5\log 6 - 5\log 5 + 6\log 7 - 6\log 6 + 14 - 7\log 7$$

$$= \boxed{14 - \log 2 - \log 3 - \log 4 - \log 5 - \log 6 - \log 7} \text{ Ans}$$

Example: (8) $y = [f(x)]$



$y = x^n$
n is odd



$y = [x^n]$; $n \neq 1$
n is odd
Step size decreases

$y = [x]$ for $n = 1$
Step size is same

Types of functions :-

1) Algebraic function ⇒ Roots of $a_0(x) + a_1(x)y + \cdots + a_n(x)y^n = 0$

2) Transcedental function
   - Exponential function
   - Trigonometric function
   - Other function

Algebraic function :- A function is said to be algebraic if it can be root of a polynomial equation $\sum_{i=0}^{n} P_i(x)y^i = 0$ in y.

Where $P_i(x)$ are polynomials in x.

Which is not algebraic is called Transcedental function.

Note :- (i) Constants functions are algebraic functions even if $f(x) = \pi$.

(ii) Non-constant algebraic function can not be periodic.

i.e. $\nexists$ no T s.t. $f(x+T) = f(x)$, $T \in \mathbb{R}$

V.Imp (iii) $f(x) = \{x\} = x - [x]$ Neither algebraic nor trigonometric because it has period 1.

Periodic function :- A function $y = f(x)$ is said to be periodic, if $f(x+T) = f(x)$ $\forall$ $x \in \mathcal{D}$; for some $T \neq 0$ Then the smallest such +ve 'T' is called fundamental period or simply period.

Note :- (i) Constant functions are periodic function without period.

(ii) Algebraic functions can not be periodic unless they are constant.

If $f(x+T) = f(x)$

$f(x+nT) = f(x) \quad \forall\ n \in \mathbb{Z}$

(iii) Sum or difference of periodic functions may not be periodic.

(iv) Sum or difference of two non-periodic functions may be periodic.

e.g.: $f(x) = x - [x]$

$f(x+1) = x+1 - [x+1] = x+1 - [x] - 1 = x - [x]$

It has period 1.

(v) If $f(x)$ has period T then $f(bx+c)$ has period $\frac{T}{|b|}$ (period is always +ve)

e.g.:- (a) $f(x) = \sin\frac{x}{3}$

Period of $\sin x = 2\pi$

$\therefore$ Period of $\sin\frac{x}{3} = 2\pi \times 3 = 6\pi$

(b) $f(x) = \{3x\}$

Period of $\{x\} = 1$,

$\therefore$ Period of $\{3x\} = 1/3$

(c) $f(x) = \left\{\frac{x}{\pi}\right\}$

Period $= \frac{1}{\frac{1}{\pi}} = \pi$

(d) $f(x) = \sin^2 x = \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{1}{2}\cos 2x$

Period $= \pi$.

(e) $f(x) = \cos^2 x = \frac{1+\cos 2x}{2} = \frac{1}{2} + \frac{1}{2}\cos 2x$ $\therefore$ Period $= \pi$

(vii) If $f_1, f_2, \cdots f_n$ are periodic functions with periods $T_1, T_2, \cdots T_n$ and l.c.m of $T_1, T_2, \cdots T_n$ exists then their linear combination of f_i's is periodic and their l.c.m decide the period of f_i's

$$\text{l.c.m of } \left\{ \frac{p_1}{q_1}, \frac{p_2}{q_2}, \cdots \frac{p_n}{q_n} \right\} ; \; p_i, q_i \in \mathbb{N}$$

$$= \frac{\text{l.c.m } \{p_1, p_2, \cdots p_n\}}{\text{H.C.F } \{q_1, q_2, \cdots q_n\}}$$

→ l.c.m of rationals and irrationals is not defined.

→ l.c.m of irrationals exists if they all are integral multiple of same irrational.

E.g. : $2\pi, 3\pi$ l.c.m $= 6\pi$

E.g. :- $\sin 3x + \cos 2\pi$ has period 2π.

↓ Period $\frac{2\pi}{3}$ ↓ Period $\frac{2\pi}{2} = \pi$ $\frac{\text{l.c.m } (2\pi, \pi)}{\text{H.C.F } (3,1)} = 2\pi$

l.c.m of $\left\{ \frac{2\pi}{3}, \pi \right\}$ = 2π

V.Imp

⇒ $\sin^2 x + \cos^2 x = 1$ ⇒ periodic but not 2π period

→ $f(x+2\pi) = f(x)$

$g(x+3\pi) = g(x)$

$\phi(x) = f(x) + g(x)$

$\phi(x+6\pi) = f(x+6\pi) + g(x+6\pi)$

$= f(x+2\pi+2\pi+2\pi) + g(x+3\pi+3\pi$

$= f(x) + g(x)$

→ $f(x) = \sin x$ Period $= 2\pi$

$g(x) = \sin \pi x$ Period $= 2$

→ $f(x) = \sin \pi x + \{x\}$ ⇒ Periodic

Period $= 2$ ↓ Period $= 1$

→ $f(x) = \sin \pi x + \{x/2\}$ ⇒ Periodic.

Period $= 2$ ↓ Period $= 2$

J.N.U → $\sin^4 x + \cos^4 x$ ⇒ Period $= \pi/2$

→ $\sin^3 x = \frac{1}{4}(\underbrace{3\sin x}_{2\pi} - \underbrace{\sin 3x}_{2\pi/3})$

$\therefore$ Period $= (2\pi, 2\pi/3) = 2\pi$.

Equality of two functions :⇒ Two functions f_1 & f_2 are said to be equal iff

(i) f_1 & f_2 have same domain D (say)

(ii) $f_1(x) = f_2(x)$ $\forall$ $x \in D$.

e.g (i) $f_1(x) = \sqrt{x^2} = |x|$

$f_2(x) = x$,

$f_1(x) \neq f_2(x)$

(ii) $f_1(x) = \log x^2 = 2\log|x|$

$f_2(x) = 2\log x$

$\therefore f_1(x) \neq f_2(x)$

Odd and Even function :-

$f(-x) = f(x)$ → even function e.g. $\cos x$, x^2 etc.

$f(-x) = -f(x)$ → odd function e.g. $\sin x$, x^3, etc.

→ Even functions are symmetrical about y-axis

→ Even functions are not one-one if the domain contain +ve as well as -ve numbers.

Restriction of function :⇒ Domain is reduced.

$f: C \to \mathbb{R}$

$X \subset C$

$\hat{f} = f \ \forall \ x \in X$

$\hat{f}: X \to \mathbb{R}$

$\hat{f}$ is restriction of f on X

$\hat{f}$ is unique $\forall$ x.

i.e restriction is unique $\forall x$

Extension of function :⇒

$f: C \to \mathbb{R}$

X is a set s.t. $\bar{f}: C \cup X \to \mathbb{R}$

$\bar{f}(x) = f(x) \ \forall \ x \in C$

& $\bar{f}(y) \in \mathbb{R} \ \forall \ y \in X$

Extensions are not unique for any extending set $C \cup X$.

Bounded function :⇒ $f(x)$ is defined on I and

$\exists \ m, M \in \mathbb{R}$

s.t. $m \le f(x) \le M \ \forall \ x \in I$

then m, M are called a lower and upper bound of $f(x)$.

Properties:-(i) If $f(x)$ is bounded then range of $f(x)$ is the subset of an interval of finite length

i.e. $f : I \longrightarrow \mathbb{R}$

$f(I) \subseteq [m, M]$

If a = the least upper bound

b = the g.l.b.

then $f(I) \subseteq [b, a]$.

(ii) The values of a and b which are supremum and infimum of $f(x)$ need not be attained by f, if attained then they are called global maximum and global minimum values of f.

i.e supremum and infimum need not be belong to range of the function.

eg:- $\sup\{x\} = 1$

(iii) Supremum which is in the range is maximum value of the function whereas infimum which is in the range is minimum value of the function.

Example:- $S = [0, 1]$

$$f(x) = \begin{cases} \frac{1}{x} & ; x \neq 0, 1 \\ 1 & ; x = 0 \\ 0 & ; x = 1 \end{cases} \quad \Rightarrow x = \frac{1}{2}$$

This function is not bounded. If is bounded below but not bounded above. f is 1-1.

Monotonic function

Monotonic
- → Monotonic increasing ↑ (non decreasing)
  If $x_1 < x_2$; $x_1, x_2 \in I$
  $\Rightarrow f(x_1) \leq f(x_2)$
- → Monotonic decreasing ↓ (non-increasing)
  If $x_1 < x_2$; $x_1, x_2 \in I$
  $\Rightarrow f(x_1) \geq f(x_2)$

Note:- (i) Monotonic functions on closed interval are bounded
i.e a monotonic function becomes bounded if it defines on closed interval. [monotonic + closed interval $\Rightarrow$ bdd]

e.g:- $f(x) = \frac{1}{x}$; $x \in [0,1]$; not define at $x = 0$
then $\frac{1}{x}$, $x \in (0,1]$

(ii) Whenever function is monotonic we assume that domain of definition is given.

(iii) Constant function $f(x) = c$ is monotonically ↑ as well as monotonically decreasing.

e.g:- $f_1(x) = [x]$ ↑ , $f_2(x) = -[x]$ ↓

[SIGNATURE]

UNIT-4

Limit, Continuity & Differentiability of Functions

Limit :-

Def^n Let $S \subseteq \mathbb{R}$ and f is defined on S.

Further let 'a' be a limit point of S, we say the function f has limit $\ell \in \mathbb{R}$ at $x=a$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t. $|f(x)-\ell| < \varepsilon$ $\forall\ x \in S \cap (a-\delta, a+\delta) - \{a\}$

We write $\lim_{x \to a} f(x) = \ell$

'ℓ' is defined limit of the function at $x=a$.

i.e For any $\varepsilon > 0$, $\exists\ \delta > 0$ (depending upon ε)

s.t. $|f(x)-\ell| < \varepsilon$ whenever $0 < |x-a| < \delta$ in S

i.e the distance between $f(x)$ & ℓ can be made as small as we please.

i.e a real number 'ℓ' is said to be the limit of f at a limit point at $x=a$ of the domain of f if whenever x approaches to 'a' through the members of domain of f, $f(x)$ approaches to 'ℓ'.

Note :- (i) If we claim $\exists$ no such δ, then we should conclude that 'ℓ' is not limit and should not say limit does not exist.

Heine's Defⁿ:- Let f be defined on S ; $x=a$. 'a' is a limit point of S, $l \in \mathbb{R}$

$$\lim_{x \to a} f(x) = l$$

iff whenever $\langle a_n \rangle \longrightarrow a$, $a_n \in S$

$$\Rightarrow \langle f(a_n) \rangle \longrightarrow l.$$

Note:- (i) If we can find two sequences $\langle a_n \rangle$ and $\langle b_n \rangle$ in S converging to 'a' but $f(a_n)$ and $f(b_n)$ converge to different limit then we say the function has no limit at $x=a$ or limit does not exist at $x=a$.

(ii) We can say limit does not exist at $x=a$ if we can find a sequence $\langle a_n \rangle$ in S s.t. $\langle a_n \rangle \longrightarrow a$ and $f(a_n)$ does not converge at all.

e.g:- (i) $sig(x) = \frac{|x|}{x}$

$\langle a_n \rangle = \langle \frac{1}{n} \rangle$, $\langle b_n \rangle = \langle -\frac{1}{n} \rangle$

$\langle a_n \rangle \to 0$

$f(a_n) = f(1/n) = 1 \longrightarrow 1$

$f(b_n) = f(-1/n) = -1 \longrightarrow -1$

e.g: (ii) $f(x) = \sin \frac{1}{x}$

domain $\mathbb{R} - \{0\} = 0$ is l.p.

$a_n = 1/n$

$\langle a_n \rangle \to 0$

$f(a_n) = \sin \frac{1}{1/n} = \sin n$

limit does not exist.

e.g. : (iii) $f(x) = \sin 1/x \; ; x \neq 0$

$$a_n = \frac{1}{n\pi} \to 0$$

$$b_n = \frac{1}{(4n-3)\pi/2} \to 0$$

$$f(a_n) = \sin n\pi = 0 \to 0$$

$$f(b_n) = \sin (4n-3)\pi/2 = 1 \to 1$$

$\therefore$ limit does not exist.

IIIrd definition of limit.

General Principal for Existence Of Limit (GPEL)

(Cauchy definition)

We say $\lim_{x \to a} f(x)$ exists if for any $\varepsilon > 0$, $\exists\, \delta > 0$

s.t $|f(x_1) - f(x_2)| < \varepsilon \quad \forall\, x_1, x_2 \in S \cap (a-\delta, a+\delta) - \{a\}$

$\to$ $x_1, x_2 \in S \cap (a-\delta, a+\delta) - \{a\}$

s.t $|f(x_1) - f(x_2)| \not< \varepsilon$

$\Rightarrow$ limit does not exist.

e.g. : $x_1, x_2 \in (0-\delta, 0+\delta)$

$f(x) = \sin 1/x$

$$x_1 = \frac{1}{n\pi/2}, \quad x_2 = \frac{1}{(n+1)\pi/2}$$

$$|f(x_1) - f(x_2)| = |\pm 1| = 1$$

$\Rightarrow$ limit does not exist.

Note :- This method is rather useful to disclaim that limit does not exist.

e.g.: Dirichlet function.

$$f(x) = \begin{cases} 1 & ; x \in \mathbb{Q} \\ -1 & ; x \in \mathbb{Q}^c \end{cases}$$

Take $a \in \mathbb{R}$ then, for any $\delta > 0$, $\exists\ x_1, x_2$ s.t. $x_1 \in \mathbb{Q}$, $x_2 \in \mathbb{Q}^c$ and $x_1, x_2 \in (a-\delta, a+\delta)$

$$|f(x_1) - f(x_2)| = |1+1| = 2$$

$\Rightarrow$ limit does not exist at any real number.
i.e limit does not exist ~~every~~ anywhere in $\mathbb{R}$.

e.g.:- $a_n \in (a - 1/n,\ a + 1/n)$

s.t. $a_n \in \mathbb{Q}$ if n is odd
$\in \mathbb{Q}^c$ if n is even

$\langle a_n \rangle \longrightarrow a$

$$f(a_n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ -1 & \text{if } n \text{ is even} \end{cases}$$

$\Rightarrow$ limit does not exist.

e.g.:- $a_n, b_n \in (a - 1/n,\ a + 1/n)$

$a_n \in \mathbb{Q}$

$b_n \in \mathbb{Q}^c$

$\langle a_n \rangle \longrightarrow a$

$\langle b_n \rangle \longrightarrow a$

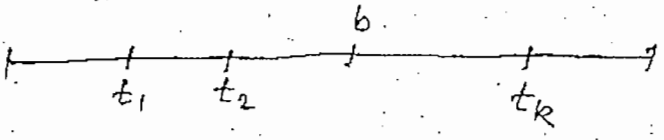
$\rightarrow \frac{1}{n} < \varepsilon, \ n > m$ $\quad \frac{1}{n} < .125 \quad n > 8 = m$

Now in $[0,1]$ the number of the form p/q will have $p \leq q$ and these are only finite number of rationals with denominator less than or equal to m.

Viz $t_1, t_2, \ldots, t_k$

Where $t_i = \frac{p_i}{q_i}$, $q_i \leq m$

Let $b \in [0,1]$



Define $\delta_i = |t_i - b|$

If $\delta = \min\{\delta_1, \delta_2, \ldots \delta_k\}$

then $t_i \notin (b-\delta, b+\delta)$.

$\rightarrow$ $$f(x) = \begin{cases} 1/q & ; \ x = p/q \text{ in } [0,1] \\ 0 & ; \ x = 0 \text{ or } x \in [0,1] \cap Q^c \end{cases}$$

This function is called Riemann function or Thoamy function or Popcorn function.

Let $l = 0$

$|f(x) - 0| = |f(x)| < \varepsilon \ ; \ x \in (b-\delta, b+\delta)$

$|f(x-0)| = |f(x)| = 1/q < q \ ; \ q > m$

$< \varepsilon$

In Riemann function limit exists everywhere in $[0,1]$. Hence limit exist at $x = 0$. It is continuous at irrationals and discontinuous at rationals.

→ $f(x) = \begin{cases} p \sin 1/q & ; x = p/q \text{ in } [0,1] \\ x & ; x \in Q^c \cap [0,1] \text{ or } x = 0 \end{cases}$

$a_n, b_n \in (b - 1/n, b + 1/n)$

$p_n/q_n = a_n \in Q, \quad b_n \in Q^c$

$f(a_n) = p_n \sin 1/q_n = \lim_{n \to \infty} p_n/q_n \times \frac{\sin 1/q_n}{1/q_n} = b$

$f(b_n) = \lim b_n = b$

It is continuous at irrational and discontinuous at rational.

Left Hand Limit (L.H.L)

Defn:- We say a function $y = f(x)$ has L.H.L at $x = a$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$$|f(x_1) - f(x_2)| < \varepsilon$$

$\forall\ x_1, x_2 \in (a - \delta, a)$ in S.

→ If $\langle a_n \rangle \to a$, $\langle b_n \rangle \to a$, $\langle c_n \rangle \to a$, $a_n, b_n, c_n < a$ then $\lim \langle f(a_n) \rangle = \lim \langle f(b_n) \rangle = \lim \langle f(c_n) \rangle = l$.

Right Hand Limit (R.H.L) :-

Defn:- We say a function $y = f(x)$ has R.H.L at $x = a$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $|f(x_1) - f(x_2)| < \varepsilon$ $\forall\ x_1, x_2 \in (a, a + \delta)$ in S.

eg:- $f(x) = \begin{cases} \sin 1/x & ; x > 0 \\ x & ; x \leq 0 \end{cases}$

$a_n = \frac{1}{n\pi}$

$b_n = \frac{1}{(4n-3)\pi/2}$

$f(a_n) = 0 \longrightarrow 0$

$f(b_n) = 1 \longrightarrow 1$

Function has L.H.L but not R.H.L.

$\rightarrow f(x) = \begin{cases} \sin 1/x & ; x > 0 \\ x & ; x \leq 0 \end{cases}$

L.H.L exists but R.H.L does not exist.

$\rightarrow f(x) = [x]$

L.H.L and R.H.L exists but not equal.

$\rightarrow f(x) = \begin{cases} x & ; x \geq 0 \\ \sin 1/x & ; x < 0 \end{cases}$

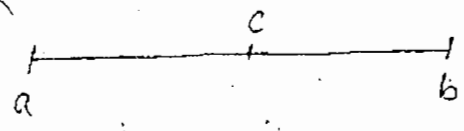
R.H.L exists but L.H.L does not exist.

$\rightarrow f(x) = \sin 1/x \quad ; x \neq 0$

R.H.L and L.H.L both does not exist.

Result:- If f is monotonic in I (open) and $c \in I$ then at c L.H.L and R.H.L both exists may not be equal.

Proof:- Without loss of generality (WLG) f is increasing.



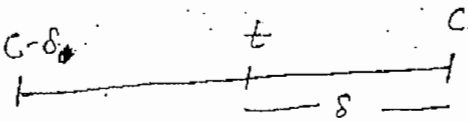
As f is increasing

$\Rightarrow$ f is bounded above in $(c-\delta, c)$

Let $l = \sup\{f(x) : x \in (c-\delta, c)\}$

i.e. for any $\varepsilon > 0$ $\exists$ $t \in (c-\delta, c)$

s.t. $f(t) > l - \varepsilon$:



$t = c - \delta$

$x \in (c-\delta, c)$ then $f(x) \geq f(t)$ because function is $\uparrow$

i.e. $f(x) \geq f(t) > l - \varepsilon$

But $f(x) < l + \varepsilon$

$l - \varepsilon < f(x) < l + \varepsilon$; $x \in (c-\delta, c)$

i.e. $|f(x) - l| < \varepsilon$; $x \in (c-\delta, c)$

$\therefore$ L.H.L exists and it is equal to the supremum of the function l.

Similarly R.H.L exists and it is equal to the infimum of the function l'.

→ $S = \{ l \in \mathbb{R} : \lim f(a_n) = l \text{ where } \langle a_n \rangle \to c \}$
Where 'c' is any fixed point. i.e. $c \in I$.

$f(x) = [x]$

$g(x) = \sin 1/x$

$T = |\{ l \in \mathbb{R} : \lim g(a_n) = l ; \text{ where } \langle a_n \rangle \to 0 \}$

S has cardinality 2

i.e. $S = \{-1, 0\}$ i.e. $|S| = 2$

$T = [-1, 1]$.

→ $t \in [-1, 1]$ then $\alpha \in \mathbb{R}$.

$\sin\alpha = t$

$\Rightarrow \sin(2n\pi + \alpha) = t$

Construct $a_n = \frac{1}{2n\pi + \alpha} \to 0$

$$g(a_n) = \sin \frac{1}{a_n} = \sin(2n\pi + \alpha) = t$$

→ $S = \{ l \in \mathbb{R} : \lim f(a_n) = l ; \langle a_n \rangle \to c \}$

Limit exists if S is singleton.

→ $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} x & ; x \in Q \\ -x & ; x \in Q^c \end{cases}$$

Limit exists only at $x = 0$ and nowhere limit exist.

$$|f(x) - 0| = \begin{cases} |x - 0| = |x - 0| < \varepsilon = \delta \\ |-x - 0| = |x - 0| < \varepsilon = \delta. \end{cases}$$

→ $f(x) = \begin{cases} x & x \in Q \\ 1-x & x \in Q^c \end{cases}$

Show that at $x = 1/2$ limit exists and equal to $1/2$.

Solution :- $a = 1/2$

$l = 1/2$

$|f(x) - 1/2| < \begin{matrix} |x - 1/2| < \varepsilon = \delta \\ |1 - x - 1/2| < \varepsilon = \delta \end{matrix}$

i.e. $|x - 1/2| < \varepsilon = \delta$

→ $f(x) = \begin{cases} 1/q & x \in p/q \text{ in } [0,1] \\ 0 & x = 0 \text{ or } x \in Q^c \cap [0,1] \end{cases}$

$a_n \in (a - 1/n, a + 1/n)$

$a_n \in Q^c \quad \langle a_n \rangle \to a$

$\langle f(a_n) \rangle \to 0$

→ $sig(\sin 1/x) = \begin{cases} 1 & ; \sin 1/x > 0 \\ 0 & ; \sin 1/x = 0 \\ -1 & ; \sin 1/x < 0 \end{cases}$

→ $f(x) = \begin{cases} x & ; x \in Q^c \\ \sqrt{\frac{1+p^2}{1+q^2}} & ; x = p/q \end{cases}$

Limit does not exist at -ve real numbers

Limit exists only at +ve real numbers and zero.

i.e. limit exists at all non-negative real numbers.

$\sqrt{\frac{1+p^2}{1+q^2}} = \left|\frac{p}{q}\right| \sqrt{\frac{1+1/p^2}{1+1/q^2}}$

$a \in R$

$a_n \in (a - 1/n, a + 1/n)$ s.t. $a_n \in Q^c$

$\langle a_n \rangle \to a$

$$\lim f(a_n) = \lim a_n = a$$

Let $b_n = \frac{p_n}{q_n} \to a$

$$f(b_n) = \sqrt{\frac{1+p_n^2}{1+q_n^2}}$$

$$= \left|\frac{p_n}{q_n}\right| \left|\frac{1+1/p_n^2}{1+1/q_n^2}\right|$$

$$= |a|$$

Some Important Results :-

(1) At any point for the monotonic function LHL and RHL both exist may not be equal.
Hence for a monotonic function set of points where limit does not exist is countable i.e. countably infinite.
i.e. set of discontinuities of monotonic function is countable.

Problem :- Find limit $f(x) = \{x\} = x - [x]$ at $x = 1$.

Solution :- R.H.L

$$\lim_{x \to 1^+} \{x\} = \lim_{h \to 0} \{1+h\} = \lim_{h \to 0} h = 0 \quad \text{limit exists.}$$

L.H.L

$$\lim_{x \to 1^-} \{x\} = \lim_{h \to 0} \{1-h\} = \lim_{h \to 0} 1-h = 1 \quad \text{limit exists}$$

R.H.L ≠ L.H.L

Limit does not exist at $x = 1$.

Problem :- $f(x) = [\sin x]$, find limit at $x = 0$

Solution :- R.H.L $\lim_{x \to 0^+} [\sin x] = \lim_{h \to 0} [\sin(0+h)] = \lim_{h \to 0} [\sin h] = 0$

L.H.L:

$\lim_{x\to 0^-} [\sin x] = \lim_{h\to 0} [\sin(0-h)] = \lim_{h\to 0} [-\sin h] = -1$

↓ value lies between 0 & 1.

R.H.L ≠ L.H.L.

⇒ At $x=0$, limit does not exist.

Problem:- $f(x) = [\cos x]$. Find limit at $x=0$

Solution:- R.H.L

value lies between 0 & 1 ↑

$\lim_{x\to 0^+} [\cos x] = \lim_{h\to 0} [\cos(0+h)] = \lim_{h\to 0} [\cos h] = 0$

L.H.L

$\lim_{x\to 0^-} [\cos x] = \lim_{h\to 0} [\cos(0-h)] = \lim_{h\to 0} [\cos h] = 0$

L.H.L = R.H.L

At $x=0$, limit exists.

Some other type examples related to function:-

Example (1): $f: \mathbb{R}\to\mathbb{R}$, $f(x) = (x^2+1)^{2007}$

-ve and +ve elements have same images, so not 1-1.

-ve elements have no pre-image, so not onto.

So f is neither 1-1 nor onto.

Hence f is not bijective.

Example(2): $f(z) = \left(1+\frac{1}{i}\right)^7 =$ constant

Constant function is neither 1-1 nor onto.

Hence f is many-one and into.

Continuity

Definition:- Let $a \in S$, we say f is continuous at 'a' where f is defined on S if 'a' is an isolated point of S or if 'a' is limit point of S then

$$\lim_{x \to a} f(x) = f(a)$$

→ There are following three cases when $f(x)$ is discontinuous at $x=a$:

(1) $\lim_{x \to a} f(x)$ does not exist

e.g.:- (i) $f(x) = \{x\}$ at $x = 1$

(ii) $f(x) = \sin 1/x$ at $x = 0$.

(2) $\lim_{x \to a} f(x)$ exists but $f(x)$ is not defined at $x=a$

e.g.:- (i) $f(x) = \sin x \; ; \; x \neq \pi/2$

$f(x)$ has discontinuity at $x = \pi/2$

(3) $\lim_{x \to a} f(x)$ exists but unequal to $f(a)$.

e.g.:- $$f(x) = \begin{cases} \frac{\sin x}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$\lim_{x \to 0} \frac{\sin x}{x} = 1$

But $f(0) = 0$

Some examples:-

1). $f(x) = \cos x \; ; \; x \neq 1$ — discontinuous at $x = 1$

2). $f(x) = \begin{cases} \sin x/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ — discontinuous at $x = 0$

3). $f(x) = \{x\} \; ; \; x \in [0, 3]$ — discontinuous at $x = 0, 1, 2, 3$.

4): $f(x) = \begin{cases} 1 & ; x \in Q \\ -1 & ; x \in Q^c \end{cases}$

is called Dirichlet function.

It is no where continuous because limit does not exist.

5). $f(x) = \sin x \; ; \; x \in (0,1) \cup \{2\}$

This function is continuous at 2.

Note :- (i) If f is not continuous at $x = a$ then we say f is discontinuous.

(ii) If limit points of S are not member of S then f is assumed to be discontinuous there at.

e.g. : $f(x) = \sin x \; ; \; x \neq 1$

Domain $= \mathbb{R} - \{1\}$

At $x = 1$, f is discontinuous.

(iii) At isolated points, $f(x)$ is said to be continuous if it is defined there at.

e.g. :- $f(x)$ is defined on $[0,1] \cup \{3\}$ s.t. $f(x) = \sin x$

(iv) $f(x) = \sin^{-1} x$ on $[0,1] \cup \{3\}$.

$f(x)$ is discontinuous at 3.

Second definition of Continuity :-

We say $f(x)$ is continuous at $x = a$ iff $\langle a_n \rangle \to a$

$\Rightarrow f(a_n) \to f(a)$.

→ This definition is best for proving function is discontinuous.

Types of discontinuity

1). **Removable discontinuity** :- If $\lim_{x \to a} f(x)$ exists but either $f(x)$ is not defined at $x=a$ or it is defined and $\lim_{x \to a} f(x) \neq f(a)$ then we say that the function has removable discontinuity at that point.

i.e. If $\lim_{x \to a} f(x)$ exists but not equal to value of the function then we say f has removable discontinuity.

i.e (i) $a \notin S$

(ii) $\lim_{x \to a} f(x) \neq f(a)$.

e.g :- (i) $f(x) = \frac{\sin x}{x}$ at $x=0$ Domain $(R - \{0\})$

(ii) $f(x) = \begin{cases} \frac{\sin x}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ discontinuity at $x=0$

(iii) $f(x) = \sin x$; $x \neq 1$ discontinuity at $x=1$.

(iv) $f(x) = x$; $x \in (0,1)$ discontinuity at $x=0$ & $x=1$.

This type of discontinuity can be removed by defining a new function g as

$$g(x) = \begin{cases} f(a) & ; \text{ if } x \neq a \\ \lim_{x \to a} f(x) & ; \text{ if } x = a \end{cases}$$

Then g is continuous at 'a'.

It is also called Simple discontinuity.

Cont $\Rightarrow$ Diff

Continuity always check at domain member.

2) **Discontinuity of first kind or (Jump Discontinuity)**

We say a function $f(x)$ has discontinuity of first kind (or jump discontinuity) at $x=a$ if at this point L.H.L and R.H.L both exists but not equal.

→ f is said to have a discontinuity of the 1st kind from the left at a if L.H.L exists but is not equal to $f(a)$.

→ f is said to have a discontinuity of the 1st kind from the right to a if R.H.L exists but is not equal to $f(a)$.

OR

L.H.L

$\lim_{x \to a^-} f(x)$ exists

if for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $\forall\ x_1, x_2 \in (a-\delta, a)$

$|f(x_1) - f(x_2)| < \varepsilon$

R.H.L.

$\lim_{x \to a^+} f(x)$ exists

if for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $\forall\ x_1, x_2 \in (a, a+\delta)$

$|f(x_1) - f(x_2)| < \varepsilon$.

Jump at $(x=a) = \left| \lim_{x \to a^-} f(x) - \lim_{x \to a^+} f(x) \right|$

$\sum \frac{1}{n^p}$ $(p>1)$

e.g.: (i) $f(x) = [x]$; $x = 1$

(ii) $f(x) = \{x\}$; $x = 1$

(iii) $f(x) = sig(x)$; $x = 0$

(iv) $f(x) = \begin{cases} x & ; x \geq 0 \\ \sin\frac{1}{x} & ; x < 0 \end{cases}$ } not jump kind discontinuity. It is end kind discontinuity.

$\longrightarrow f(x) = [e^x]$

$\log e^1, \log e^2, \ldots \log e^n$ (discontinuous)

$\log e^n = \log e^{n-1}$

$Jx = \log n = \log_e\left(\frac{n}{n-1}\right) = \log_e\left(1 + \frac{1}{n-1}\right)$

↓ jump at $x = \log n$.

Discontinuity of second kind or Mixed discontinuity

If any of RHL or LHL or both does not exist at $x = a$. We say at $x = a$ function has 2nd kind discontinuity.

e.g.:- (i) $f(x) = \begin{cases} x & ; x \geq 0 \\ \frac{\sin x}{x} & ; x < 0 \end{cases}$

(ii) Dirichlet function

$$f(x) = \begin{cases} 1 & ; x \in Q \\ -1 & ; x \in Q^c \end{cases}$$

Let $a \in R$, $x_1, x_2 \in (a, a+\delta)$ s.t. $x_1 \in Q$, $x_2 \in Q^c$

$|f(x_1) - f(x_2)| = 2$

$\Rightarrow$ R.H.L does not exist

Similarly, LHL does not exist.

$\Rightarrow$ Limit does not exist at any real number.

(iii) $x^2 sig(\cos x) = \begin{cases} x^2 ; & \cos x > 0 \\ 0 ; & \cos x = 0 \\ -x^2 ; & \cos x < 0 \end{cases}$

Result:- Monotonic function can have only jump kind of discontinuity or first kind of discontinuity over domain of the function.

Result:- The number of discontinuities of monotonic function is similar to the set of disjoint intervals.

i.e. The set of discontinuity of monotonic function is always countable.

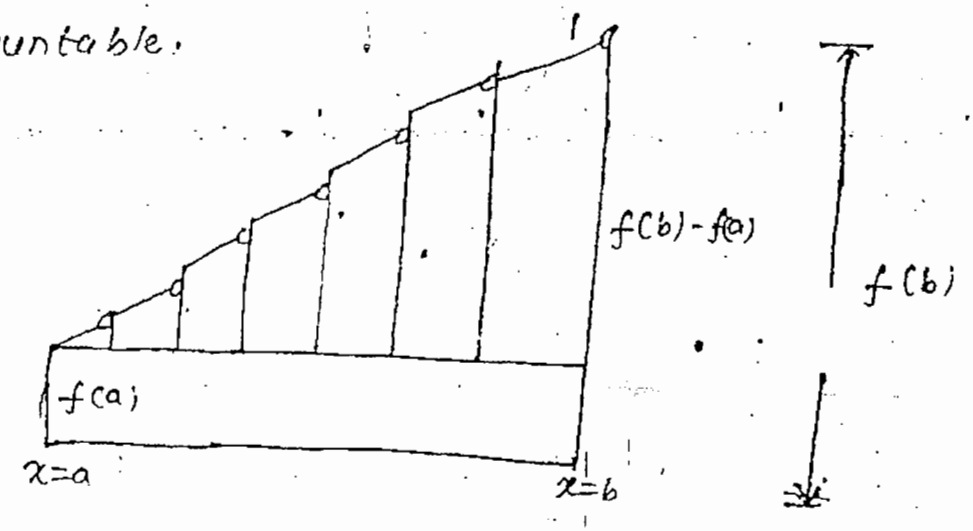
Proof:- $S = \{$ set of jump of $f(x)\}$

Let $\inf S = \delta$

then for any δ, $\exists\ m$ s.t. $\frac{f(b)-f(a)}{n} < \delta$, $n > m$,

then function can have at most n jump of this size.

Hence the set of discontinuity of monotonic function is countable.



Infinite limit:- We say $\lim_{x\to a} f(x)$ exists infinitely if for any $M>0$, $\exists\ \delta>0$ s.t. $|f(x)|>M$ whenever $0<|x-a|<\delta$.

→ If $f(x)>M \Rightarrow \lim_{x\to a} f(x) = +\infty$

→ If $f(x)<-M \Rightarrow \lim_{x\to a} f(x) = -\infty$

Note:- nbd of $\infty = |x|>lR = (-\infty,\infty) = lR$

Limit at ∞:- $\lim_{x\to\infty} f(x)$ exists finitely and equal to l if for any $\varepsilon>0$; $\exists\ lR>0$ s.t. $|f(x)-l|<\varepsilon$ whenever $|x|>lR$.

Note:- $\lim_{x\to\infty} x^2\phi(x)$ does not exist if $\phi(x)$ oscillates infinitely many times in the nbd of ∞.

Proof:- Let $f(x)$ be increasing.

WLG (Without loss of generality)

Then $x_1 < x_2$

$\Rightarrow f(x_1) \le f(x_2)\ \forall\ x_1, x_2 \in (a,b)$

Let $c \in (a,b)$, Let $x \in (c-\delta, c)$

i.e. for any $\varepsilon>0$, $l-\varepsilon$ fails to be supremum

i.e. $\exists\ y \in (c-\delta, c)$ s.t. $f(y) < l-\varepsilon$

Now as $f(x)$ is increasing, $f(x) > f(y)\ \forall\ x \in (y,c)$

In fact $f(x) > l-\varepsilon\ \forall\ x \in (y,c)$

Define $\delta = c-y$

$f(x) > l-\varepsilon\ \forall\ x \in (c-\delta, c)$

and $f(x) < l+\varepsilon\ \forall\ x \in (c-\delta, c)$

$f(x) \in (l-\varepsilon, l+\varepsilon) \quad \forall\, x \in (c-\delta, c)$

$\Rightarrow \quad |f(x) - l| < \varepsilon \quad \forall\, x \in (c-\delta, c)$

Hence $\boxed{\lim_{x \to c^-} f(x) = \sup_{x \in (c-\delta, c)} f(x)}$

Similarly, $\boxed{\lim_{x \to c^+} f(x) = \inf_{x \in (c, c+\delta)} f(x)}$ Hence proved ##

→ Simple discontinuity → Removable discontinuity
→ 1st kind discontinuity or Jump discontinuity.

Properties Of Continuous Functions:

	continuous
① → $\sin x$	$\mathbb{R}$
→ $\cos x$	$\mathbb{R}$
→ $P(x) = \sum a_i x^i$	$\mathbb{R}$
→ e^x	$\mathbb{R}$
→ $\log e^x$	$(0, \infty)$.
→ $\frac{p(x)}{q(x)}$	$\mathbb{R} - \{x : q(x) = 0\}$
→ $[f(x)]$	$\mathbb{R} - \{x : f(x) \in \mathbb{Z}\}$
→ $\{f(x)\}$	$\mathbb{R} - \{x : f(x) \in \mathbb{Z}\}$
→ $f+g,\ f-g$	
→ $f \cdot g,\ f/g ; g \neq 0$	
→ $f \circ g$	
→ $\lvert f \rvert,\ \lvert g \rvert$, converse is not true	

→ $\max\{f, g\}$

↳ $\min\{f, g\}$

→ $h(x) = x - [x]$
↓ ↳ discontinuous
continuous

→ Sum and difference of continuous and discontinuous function is equal to discontinuous function.

(2). Continuity of a function is defined at a point i.e it is a property of a function w.r.t. a point of given domain.

(3) If $f(x)$ is continuous but $g(x)$ is not continuous at some point $x=a$ then $f(x)+g(x)$ and $f(x)-g(x)$ fail to be continuity.

(4). $(f \circ g)(x)$ may be continuous even if one may be discontinuous or two may be discontinuous

E.g :- (i) $f(x) = \begin{cases} 1 & ; x \in Q \\ -1 & ; x \in Q^c \end{cases}$

$(f \circ f)(x) = 1 \ \forall x$ is every where continuous.

and $g(x) = \begin{cases} x & ; x \in Q \\ 1-x & ; x \in Q^c \end{cases}$

$g(x)$ is continuous at $x = 1/2$ only

NOW, $(g \circ g)(x) = g(x) = x$

$= g(1-x) = 1-(1-x) = x$

V.Imp

Remark :- $g(x) = \begin{cases} p(x) & ; x \in Q \\ q(x) & ; x \in Q^c \end{cases}$

This function $g(x)$ is continuous only at the roots

$p(x) - q(x) = 0$ where $p(x)$ and $q(x)$ are continuous functions.

e.g.:- (i) $g(x) = \begin{cases} x^2+6 & ; x \in Q \\ 5x & ; x \in Q^c \end{cases}$

$p(x) - q(x) = 0 \Rightarrow x^2 + 6 - 5x = 0 \Rightarrow x = 2, 3.$

The $g(x)$ is continuous at $x = 2, 3$ only.

(ii) $g(x) = \begin{cases} \sin x & ; x \in Q \\ \cos x & ; x \in Q^c \end{cases}$

Now, $\sin x - \cos x = 0 \Rightarrow \tan x = 1 \Rightarrow x = n\pi + \frac{\pi}{4}, n \in Z$

i.e. $g(x)$ is continuous at $x = n\pi + \frac{\pi}{4}$, countably infinite.

(iii) $g(x) = \begin{cases} \sin x & ; x \in Q \\ \cosh x = \frac{e^x + e^{-x}}{2} & ; x \in Q^c \end{cases}$

Here $-1 \le \sin x \le 1$ A.M >, G.M

$\cosh x = \frac{e^x + e^{-x}}{2} \ge 1$

$\sin x - \cosh x = 0$; such x does not exist.

i.e. $g(x)$ is not continuous.

(iv) $g(x) = \begin{cases} \cos x & ; x \in Q \\ \cosh x = \frac{e^x + e^{-x}}{2} & ; x \in Q^c \end{cases}$

Now, $\cos x - \cosh x = 0$ By graph.

$\boxed{x = 0}$

i.e. $g(x)$ is continuous only at $x = 0$

(v) $g(x) = \begin{cases} e^{1/x} + e^{-1/x} & ; x \in Q \\ \cos x & ; x \in Q^c \end{cases}$

Then $g(x)$ is not continuous.

→ $g(x) = 1/x$

$f(x) = \sin x$

$(f \circ g)(x) = f(1/x) = \sin 1/x$

It is continuous throughout $\mathbb{R}$ except at $x=0$.

→ If f is continuous at $x=a$

⇒ for any $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$.

→ Let $f(a) > 0$

Put $\varepsilon = \frac{1}{2} f(a)$

$\exists\ \delta > 0$ s.t. $\forall\ x \in (a-\delta, a+\delta)$

$$|f(x) - f(a)| < \tfrac{1}{2} f(a)$$

$$-\tfrac{1}{2} f(a) < f(x) - f(a) < \tfrac{1}{2} f(a)$$

$$\Rightarrow \tfrac{1}{2} f(a) < f(x) < \tfrac{3}{2} f(a) \quad ; \forall\ x \in (a-\delta, a+\delta)$$

→ $f(x) = x \quad ; x \in (0,1)$

Function f has discontinuity at $x=0$ & $x=1$.

But f has continuity throughout the domain $(0,1)$

i.e. f has removable discontinuity at $x=0$ & $x=1$.

→ $$f(x) = \begin{cases} p \sin 1/q & ; x = p/q \text{ in } (0,1) \\ -x & ; x = 0 \text{ or } x \in \mathbb{Q}^c \cap (0,1) \end{cases}$$

Removable discontinuity at $x=0$ & $x=1$

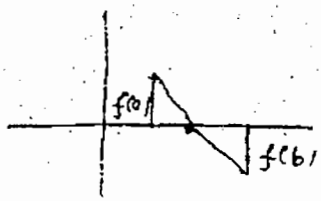
Result:- If f is continuous on I and $a \in I$, If $f(a) \neq 0$ then $\exists$ a nbd of 'a' in which $f(x)$ retains its sign.

i.e. if $f(a)$ is +ve then $\exists\ \delta > 0$ s.t. $f(x)$ is +ve $\forall\ x \in (a-\delta, a+\delta)$.

Result:- If f is continuous on I and $a, b \in I$ and $f(a) \cdot f(b) < 0$ then $\exists\ c \in (a,b)$ s.t. $f(c) = 0$.

Proof:- Let $f(a) < 0$, $f(b) > 0$

$S = \{x \in [a,b] : f(x) < 0\}$ then



$a \in S$ and $S \subset [a,b]$

Let $\sup S = c$

$f(c) < 0$ ——— (i)

$\exists\ \delta > 0$ s.t. $f(x) < 0\ \forall\ x \in (c-\delta, c+\delta)$

$\Rightarrow f(c + \delta/2) < 0$.

$f(c) > 0$ ——— (ii)

$\exists\ \delta > 0$ s.t. $f(x) > 0\ \forall\ x \in (c-\delta, c+\delta)$

$f(x) > 0$ for some $x < c$

$\Rightarrow \sup S \neq c$

By equations (i) & (ii), we have

$\boxed{f(c) = 0}$ Hence proved

Result (i):- If f is continuous on $[a,b]$ then it attains its supremum and infimum on the interval i.e, they have max^m and min^m.

eg:- $(1,2)$; $f(x) = 1/x$ → It does not have max^m & min^m but it has supremum & infimum.

Result ② If f is continuous on closed interval $[a,b]$ then it is bounded.

i.e Continuity + closed interval $\Rightarrow$ boundedness.

Proof of ① $\sup_{x\in[a,b]} f(x) = K$

$f(x) \leq K \quad \forall\, x \in [a,b]$

Let $f(x) \neq K$ for any $x \in [a,b]$

Construct,

$$g(x) = \frac{1}{K-f(x)} < M \quad ; \quad K - f(x) > \frac{1}{M}$$

$$f(x) < K - \frac{1}{M} \quad \forall\, x \in [a,b]$$

$$\sup f(x) \leq K$$

Hence proved.

Proof of ② :- Let $f(x)$ is unbounded. then for any $n \in \mathbb{N}$ $\exists\, a_n \in (a,b)$ s.t. $f(a_n) > n$

$\langle a_n \rangle$ is bold as $a_n \in [a,b]$ $\forall\, n \in \mathbb{N}$

$\Rightarrow \langle a_n \rangle$ has a limit point l (say), $l \in [a,b]$

$\exists\, \langle a_{n_k} \rangle$ s.t. $\lim a_{n_k} = l$

But $f(a_{n_k}) > n_k$

$\langle a_{n_k} \rangle \longrightarrow l$

$\langle f(a_{n_k}) \rangle \longrightarrow$ unbounded

$\Rightarrow$ By Heines theorem, $f(x)$ is not continuous at $x = l$ which is a contradiction.

Hence our assumption is wrong.

Hence proved

→ If $\sup f(x) = M$, $x \in [a,b]$ then $\exists\ t \in [a,b]$ s.t. $f(t) = M$.

→ Continuity of a function is a pointbase property.

→ V. Imp Strictly increasing $\Rightarrow$ 1:1 but not conversely.

→ A function is identically zero which has no zero.

→ $$f(x) = \begin{cases} 1/q & ; \ x = p/q \text{ in } [0,1] \rightarrow \text{discontinuous} \\ 0 & ; \ x = 0 \ ; \ x \in Q^c \cap [0,1] \rightarrow \text{continuous} \end{cases}$$

$a \in Q$

$f(a) \neq 0$

$a_n \in (a - 1/n, a + 1/n) \cap Q^c$

$f(a_n) = 0$

$f(a_n) \to 0 \neq f(a)$

This function is called Riemann function.
This function is discontinuous on rationals and continuous on irrationals.

→ Removable & Jump discontinuities are referred as simple discontinuities and other are non simple i.e. 2nd kind of discontinuity is non simple discontinuity.

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9mb

Fixed Point :- If $f(x)$ is defined on $[a,b]$ and for some $c \in [a,b]$ s.t. $f(c) = c$. then c is called a fixed point of $f(x)$. In fact, the fixed points of $f(x)$ are the points of intersection of the curve $y = f(x)$ and $y = x$ i.e. the roots of $f(x) - x = 0$

9mb

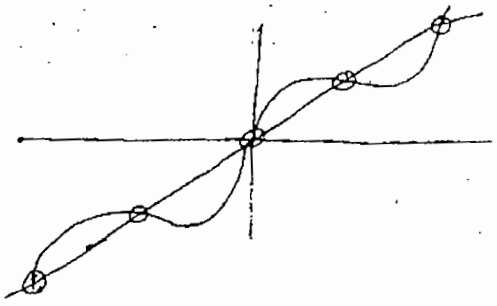
Note :- If $f : [a,b] \longrightarrow [a,b]$ (same & closed) and continuous then $\exists\ c \in [a,b]$ s.t. $f(c) = c$

i.e. then the function has a fixed point.

→ The point need not be unique.

→ $f(x) = x + \sin x$

$n\pi$ are the fixed points of this function.



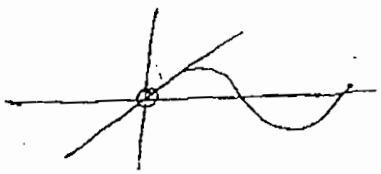
i.e. $f(x) - x = x + \sin x - x = 0$

$\Rightarrow \sin x = 0$

$\Rightarrow x = n\pi$; where $n \in Z$.

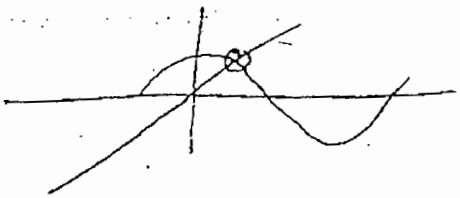
→ $f(x) = \sin x$

Only one fixed point.



→ $f(x) = \cos x$

Only one fixed point



Fixed Point Theorem

Property :- If $f : [a,b] \longrightarrow [a,b]$ and continuous then $\exists\ c \in [a,b]$ s.t. $f(c) = c$.

Proof :- Let $\phi(x) = f(x) - x$

$\phi(a) = f(a) - a \geq 0$ if $f(a) - a = 0$

$\phi(b) = f(b) - b \leq 0$ $\Rightarrow f(a) = a$

$\phi(a).\ \phi(b) \leq 0$

$\exists\ c \in [a,b]$.

$\boxed{\phi(c) = 0}$

if $f(b) - b = 0$

$\Rightarrow f(b) = b$

$\therefore$ b is fixed point.

Note:- If ϕ is 1-1 or strictly increasing or strictly decreasing then either it will have no fixed point or unique fixed point.

Intermediate Value Property (I.V.P):- If f is continuous on $[a,b]$ and let $f(a) < f(b)$ and $f(a) < k < f(b)$ then $\exists\ c \in (a,b)$ s.t. $f(c) = k$

i.e. $[f(a), f(b)] \subseteq f[a,b]$

i.e. the image of an interval under continuous function is either interval or singleton function is constant.

Proof:- $\phi(x) = f(x) - k$

$\phi(a) = f(a) - k < 0$

$\phi(b) = f(b) - k > 0$

$\Rightarrow \phi(a).\ \phi(b) < 0 \Rightarrow \exists\ c \in (a,b)$ s.t. $g(c) = 0$

$\Rightarrow f(c) - k = 0$

$\Rightarrow \boxed{f(c) = k}$ Hence proved

$\rightarrow f : [a,b] \xrightarrow{\text{continuous}} \mathbb{R}$

$m \leq f(x) \leq M$

$[m, M] = f[a,b]$

Range.

→ $f: \mathbb{R} \xrightarrow{\text{continuous}} \mathbb{Q}$ then function is constant.

Result:- Every continuous function from an interval to that set which cannot contain interval is constant.

e.g.: $f: (a,b) \xrightarrow{\text{continuous}}$ → $\mathbb{Q}$, → $\mathbb{Q}^c$, → any countable set, → any discrete set

V.Imp → For any non-constant continuous function the range is uncountable where the domain of the function is an interval.

counter example

$$S = [0,1) \cup (1,2)$$

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 2 & 1 < x < 2 \end{cases}$$

→ Range of continuous function is either singleton or uncountable.

→ $$f(x) = \begin{cases} x & 0 < x < 1 \\ 1 & x = 0 \\ 0 & x = 1 \end{cases}$$

onto function
1-1 function

– $f(0) = 1$

$f(1/2) = 1/2$ $\qquad f(c) < 3/4 < f(1)$

$\exists\, c \in (0, 1/2)$

$f(c) = 3/4$

I.V.P. does not hold.

Problem:- $f: \mathbb{R} \xrightarrow{\text{continuous}} \mathbb{R}$ then $f(\mathbb{R}) \in$?

(i) [redacted]: f is constant

(ii) f is strictly increasing.

(iii) f is strictly decreasing.

(iv) none of these

→ If $f = [a,b]$, $t_1, t_2 \in [a,b]$

$f(t_1) < \alpha < f(t_2)$ $\exists\ c \in (t_1, t_2)$ s.t. $f(c) = \alpha$.

→ If f holds IVP on $[0,1]$ then if $t_1, t_2 \in [0,1]$

$f(t_1) < \alpha < f(t_2)$ $\exists\ c \in (t_1, t_2)$ s.t. $f(c) = \alpha$.

$t_1 = 0$, $t_2 = 1/2$

$f(t_1) = 1$, $f(t_2) = 1/2$

$f(t_2) < 0.9 < f(t_1)$

→ (v.imp) One-one and onto of a function does not imply I.V.P.

Result:- If f is monotonic and it holds I.V.P. then it is continuous.

i.e. **Monotonicity + I.V.P ⇒ continuity.**

Note (i) Every strictly increasing (decreasing) function is 1-1.

(ii) 1-1 ⇒ monotonicity

but monotonicity ⇏ 1-1

E.g.: $f(x) = [x]$ is monotonic ↑ but not 1-1.

(& constant function)

Identity Theorem:- If f is continuous on $\mathbb{R}$ and f is zero at rational then it is identically zero i.e. it is continuous at every where domain i.e. $\mathbb{R}$.

OR, If $f(x)$ is continuous on $\mathbb{R}$ define on $\mathbb{R}$ and $f(x)=0 \ \forall\ x\in\mathbb{Q}$ then $f(x)\equiv 0 \ \forall\ x\in\mathbb{R}$ i.e. $f(x)$ is identically zero $\forall\ x\in\mathbb{R}$.

Proof:- Let $b\in\mathbb{Q}^c$

Construct $a_n\in\left(b-\frac{1}{n},\ b+\frac{1}{n}\right)$ s.t. $a_n\in\mathbb{Q}$

$\Rightarrow \langle a_n\rangle \longrightarrow b,\ \langle f(a_n)\rangle \longrightarrow f(b)$

$\Rightarrow \lim_{n\to\infty} f(a_n) = f(b).$

IInd form statement:- If $f(x)$ and $g(x)$ are continuous on $\mathbb{R}$ and $f(x)=g(x) \ \forall\ x\in\mathbb{Q}$, then $f(x)=g(x)$ $\forall\ x\in\mathbb{R}$.

Proof:- Define $h(x)=f(x)-g(x)$

$h(x)=0 \ \forall\ x\in\mathbb{Q}$

$\Rightarrow h(x)\equiv 0 \ \forall\ x\in\mathbb{R}$ (By Identity theorem)

$\Rightarrow f(x)=g(x) \ \forall\ x\in\mathbb{R}.$

Claim For Continuity:-

Result (9mp): 1) Every monotonic function is continuous if satisfied the I.V.P.

Result (9mp): 2) If $|f(x)|\le|x| \ \forall\ x\in\mathbb{R}$ then $f(x)$ is continuous at $x=0$.

Proof:- $f(0) \leq 0 \Rightarrow f(0) = 0$

$|f(x) - f(0)| = |f(x)| \leq |x| = |x-0| < \varepsilon$

$\varepsilon = \delta$

i.e. $|f(x) - f(0)| < \varepsilon$

whenever $|x-0| < \delta$.

Result (3):- (i) $|f(x)| \leq |x| \quad \forall\, x \in \mathbb{R}$

(ii) $f(x+y) = f(x) + f(y) \quad \forall\, x \in \mathbb{R}$

then $f(x)$ is continuous on $\mathbb{R}$.

Problem:- $|f(x)| \leq |x| \quad \forall\, x \in \mathbb{R}$

$f(x+y) = f(x) + f(y) \quad \forall\, x \in \mathbb{R}$

Then show that $f(x) = cx$, where $c = f(1)$

Proof:- Now, $f(x+y) = f(x) + f(y)$

$\Rightarrow f(0+0) = f(0) + f(0)$

$\Rightarrow f(0) = f(0) + f(0)$

$\Rightarrow \boxed{f(0) = 0}$

Now, $f(0) = f(x-x) = f(x) + f(-x)$

$\Rightarrow 0 = f(x) + f(-x)$

$\Rightarrow f(-x) = -f(x)$ ——— (1)

Now, $f(m) = f(1+1+\cdots \text{ m times})$

$= f(1) + f(1) + \cdots$ m times.

i.e $f(m) = m f(1), \quad m \in \mathbb{N}$

$\Rightarrow f(x) = x f(1) \quad \forall\, x \in \mathbb{N}$

Now, $f(-m) = -f(m)$ [By equation (1)]

$= -mf(1)$

Now, again

$f(x) = x f(1) \quad \forall\ x \in \mathbb{Z}$

$\Rightarrow f(1) = f\left(\frac{1}{m} + \frac{1}{m} + \cdots \text{ m times}\right)$

$= f\left(\frac{1}{m}\right) + f\left(\frac{1}{m}\right) + \cdots \text{ m times}$

$= m f\left(\frac{1}{m}\right)$

Now, $f(p/q) = f\left(\frac{1}{q} + \frac{1}{q} + \cdots \text{ p times}\right)$

$= f\left(\frac{1}{q}\right) + f\left(\frac{1}{q}\right) + \cdots \text{ p times}$

$= p f(1/q)$

$= p \cdot \frac{1}{q} f(1)$

i.e. $f(p/q) = p/q\, f(1)$

$\Rightarrow f(x) = x f(1) \quad \forall\ x \in \mathbb{Q}$

i.e. $f(x) = cx \quad \forall\ x \in \mathbb{Q}$ where $c = f(1)$

Hence by Identity theorem

$f(x) = cx \quad \forall\ x \in \mathbb{R}$.

Hence proved.

→ Let $b \in \mathbb{Q}^c$, $f(b) = ?$

$a_n \in \left(b - \frac{1}{n}, b + \frac{1}{n}\right)$, $a_n \in \mathbb{Q} \longrightarrow \lim_{n\to\infty} a_n = b$

$\langle a_n \rangle \longrightarrow b$, $\langle f(a_n) \rangle \longrightarrow f(b)$

$$f(b) = \lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} c\cdot a_n = c\lim_{n\to\infty} a_n = cb$$

$$\Rightarrow f(b) = cb \quad \forall\ b \in \mathbb{Q}^c$$

Problem:- If f is continuous & defined on $\mathbb{R}$ satisfying the conditions:

(i) $|f(x)| \le x \quad \forall\ x \in \mathbb{R}$

(ii) $f(x+y) = f(x)\cdot f(y)$

Solution:- Now $f\left(\frac{a}{2}+\frac{a}{2}\right) = f(a) = \left[f\left(\frac{a}{2}\right)\right]^2 \ge 0$

$$f(x) \ge 0 \quad \forall\ x \in \mathbb{R}$$

Now, $f(0+0) = f(0)\cdot f(0) \Rightarrow f(0) = [f(0)]^2$

$$\Rightarrow f(0) = 0, 1$$

Case I:- If $f(0)=0$, f is identically zero.

Case II:- If $f(0)=1$

$$f(m) = f(1+1+\cdots \text{ m times})$$
$$= f(1)\cdot f(1)\cdots \text{ m times}$$
$$= [f(1)]^m$$
$$= a^m \quad \forall\ m \in \mathbb{N} \text{ where } f(1)=a$$
$$\Rightarrow f(x) = a^x \quad \forall\ x \in \mathbb{N}$$

Now, $f(-m) = \frac{1}{f(m)} = \frac{1}{a^m} = a^{-m} \quad \forall\ m \in \mathbb{Z}$

$$\Rightarrow f(x) = a^x \quad \forall\ x \in \mathbb{Z}$$

Now, $f(0) = f(x-x) = f(x)\cdot f(-x)$

$$\Rightarrow 1 = f(x)\cdot f(-x)$$

$$\Rightarrow \boxed{f(-x) = \frac{1}{f(x)}}$$

$$f(1) = f(1/q + 1/q + \cdots \text{ q times})$$
$$= f(1/q) \cdot f(1/q) \cdots \text{ q times}$$
$$= [f(1/q)]^q$$
$$\Rightarrow f(1/q) = [f(1)]^{1/q} = a^{1/q}$$
$$\Rightarrow f(p/q) = [f(1)]^{p/q} = a^{p/q}$$
$$\Rightarrow f(x) = a^x \quad \forall\ x \in Q$$

Hence by Identity theorem

$$\boxed{f(x) = a^x \quad \forall\ x \in \mathbb{R}}$$ Hence proved.

Problem :- Discuss the nature of function define for +ve real value and continuous there at and satisfy the condition

$f(x \cdot y) = f(x) + f(y) \quad \forall\ x \in (0, \infty)$

find $f(x) = ?$

Solution :- $f(x) = \log x \quad \forall\ x \in (0, \infty)$

Solve by above example.

Problem :- If f is continuous on $\mathbb{R}$ and any of the following conditions are satisfied then f is constant

(i) $f(x^2) = f(x) \quad \forall\ x \in \mathbb{R}$.

(ii) $f(x) = f(2x+1) \quad \forall\ x \in \mathbb{R}$

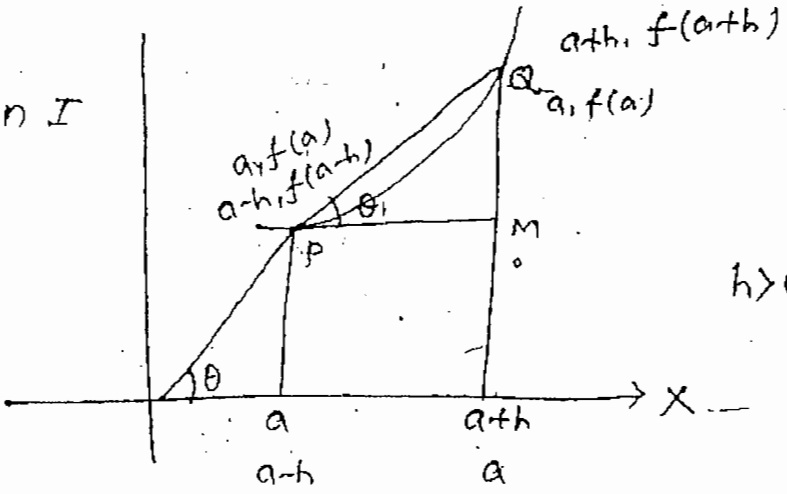
(iii) $f(x) = f(mx) \quad \forall\ x \in \mathbb{R}, m \neq 1$

m is a fixed real number not equal to 1.

Differentiability :-

$y = f(x)$ is defined on I

$x = a \in I$



Right Hand Derivative (R.H.D) at $x=a$:-

In ΔPMQ

$$\frac{QM}{PM} = \tan\theta$$

$$\Rightarrow \frac{f(a+h)-f(a)}{h} = \tan\theta$$

When $Q \to P$ i.e. $h \to 0$ $\theta \to \alpha$ (say if exists)

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \tan\alpha = f'(a^+) = R.H.D$$

and R.H.D is called the slope of the tangent to the curve $y = f(x)$ at $x=a$ from right.

Left Hand Derivative (L.H.D) at $x=a$:-

In ΔPMQ

$$\frac{QM}{PM} = \tan\theta$$

$$\Rightarrow \frac{f(a)-f(a-h)}{h} = \tan\theta$$

or, $$\frac{f(a-h)-f(a)}{-h} = \tan\theta$$

When $P \to Q$ i.e. $h \to 0$, $\theta \to \beta$ (say if exists)

then $\lim_{h \to 0} \frac{f(a-h)-f(a)}{-h} = \tan\beta = f'(a^-) = L.H.D$

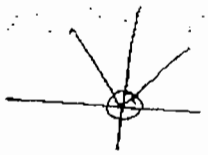
and L.H.D is called the slope of the tangent to the curve $y=f(x)$ at $x=a$ from left.

We can define differentiability

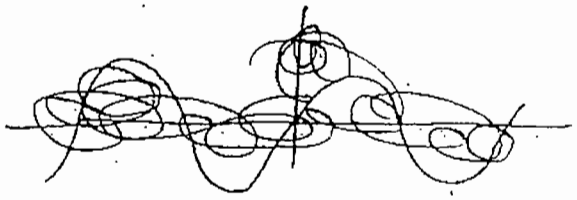
Definition (1):- A function $y=f(x)$ is said to be differentiable at $x=a$ if tangent at $x=a$ exists and unique

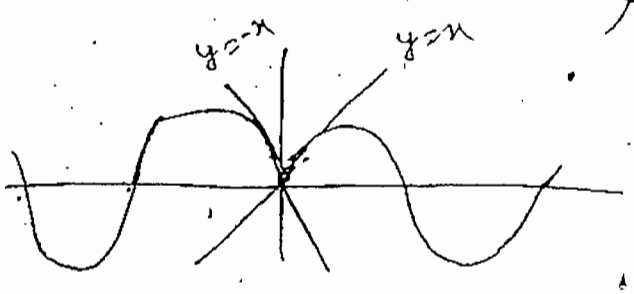
i.e. $R.H.D = L.H.D$

e.g.:- (i) $y=|x|$ is not differentiable at $x=0$



(ii) $y = \sin|x|$ is not differentiable at $x=0$.





Definition (2):- A function $y=f(x)$ is said to be differentiable at $x=a$

if $\lim_{h \to 0} \phi_1(h) = \lim_{h \to 0} \phi_2(h) = f'(a)$

Where $\phi_1(h) = \frac{f(a+h)-f(a)}{h}$

and $\phi_2(h) = \frac{f(a-h)-f(a)}{-h}$

and this limit is called derivative at that point and denoted by $f'(a)$.

Definition (3):- A function $y=f(x)$ is said to be differentiable on the closed interval $[a,b]$ if it is differentiable at every point of (a,b) and R.H.D at $x=a$ and L.H.D at $x=b$ exists finitely.

Definition (4):- A function $y=f(x)$ is said to be differentiable on open interval I if $\forall\, x, y \in I$

$\lim_{y\to x} \frac{f(y)-f(x)}{y-x}$ exists finitely $= f'(x)$

(i) $y=x+h$
(ii) $y=x-h$

Definition (5):- A function $y=f(x)$ is said to be differentiable at $x=a$ if $\exists$ a constant l (say) depending on the function $y=f(x)$ and the point $x=a$ both. s.t. $f(a+h)-f(a)-hl = h\phi(h)$

where $\lim_{h\to 0} \phi(h)=0$.

Proof:- If $f(x)$ is differentiable at $x=a$

$\exists\ \phi(x)$ s.t.

$$\frac{f(a+h)-f(a)}{h} = l + \phi(h) \quad \text{where } \lim_{h\to 0}\phi(h)=0$$

$$\Rightarrow f(a+h)-f(a)-hl = h\phi(h),$$

where, $\lim_{h\to 0}\phi(h)=0$.

Properties:-

1). If $f(x)$ is differentiable on (a,b) and $c \in (a,b)$ then we say $f(x)$ is increasing mode at c, if $f'(c) > 0$
if decreasing mode at c, if $f'(c) < 0$
if stationary mode at c, if $f'(c) = 0$.

2). $f(x)$ is in stationary mode at $x = c$ if $\underline{f'(c) = 0}$ and we say the roots $f'(x) = 0$ are stationary points of $y = f(x)$

3). All the stationary points, the tangent to the curve are parallel to x-axis and conversely.
x is stationary point $\Leftrightarrow f'(x) = 0$

4). If $f'(x) > 0 \ \forall \ x \in I$ then $f(x)$ is strictly increasing. Hence one-one. If $f'(x) < 0 \ \forall \ x \in I$ then $f(x)$ is strictly decreasing. Hence one-one.
If $f'(x) = 0 \ \forall \ x \in I$ then $f(x)$ is constant.

Imp **e.g.**: (i) $y = f(x)$, $x \in I$
s.t. $\forall \ x, y \in I$
$|f(x) - f(y)| \leq |x-y|^{1+t}$, $t > 0$
find $f(x) = ?$

Solution:- Since $|f(x) - f(y)| \leq |x-y|^{1+t}$; $t > 0$

Now, $\left| \frac{f(x) - f(y)}{x-y} \right| \leq (x-y)^t < \varepsilon$

$\Rightarrow \left| \frac{f(x) - f(y)}{x-y} - 0 \right| < \varepsilon$ whenever $|x-y| < \varepsilon^{1/t} = \delta$
$|x-y| < \delta$

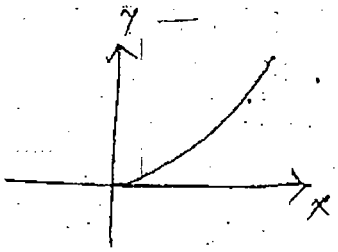
$\Rightarrow \lim_{x \to y} \frac{f(x)-f(y)}{x-y} = 0 = f'(y) \Rightarrow$ i.e function is constant.

Eg (2) $f(x) = x(x+|x|)$

$$f(x) = \begin{cases} 0 & ; x \le 0 \\ 2x^2 & ; x > 0 \end{cases}$$

$L.H.D = \lim_{h \to 0} \frac{f(0-h)-f(0)}{-h} = 0$

$R.H.D = \lim_{h \to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0} \frac{2h^2-0}{h} = 0$



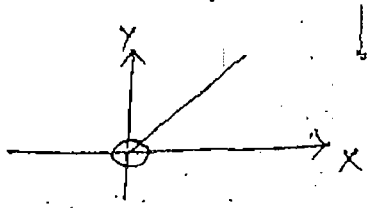
i.e. $L.H.D = R.H.D$

Hence $f(x)$ is differentiable at $x=0$

i.e. $f(x)$ is differentiable at every where.

i.e. $f'(0) = 0$

Now, $f'(x) = \begin{cases} 0 & ; x \le 0 \\ 4x & ; x > 0 \end{cases}$



$L.H.D \neq R.H.D$ at $x=0$

Hence $f'(x)$ is differentiable with except $x=0$, and continuous every where.

$f'(x) > 0 \quad \forall \ x \in (0, \infty) \Rightarrow f(x)$ is increasing.

$f'(x) = 0 \quad \forall \ x \in (-\infty, 0) \Rightarrow f(x)$ is a constant function.

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Note:- (i) $f^n(x) > 0 \ \forall x \in I$

$\Rightarrow f^{n-1}(x)$ is increasing.

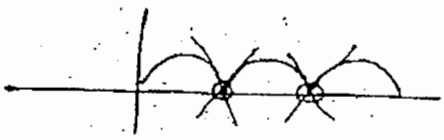
(ii) $f^n(x) < 0 \ \forall x \in I$

$\Rightarrow f^{n-1}(x)$ is decreasing.

(iii) $f^n(x) = 0 \ \forall x \in I$

$\Rightarrow f^{n-1}(x)$ is constant.

e.g.: $y = |\sin x|$



y is not differentiable at $x = \pi, 2\pi, 3\pi, \ldots$
$-\pi, -2\pi, -3\pi, \ldots$

Local Extrema

Local Maxima:- Let $y = f(x)$ is defined $\forall x \in I$ and $c \in I$, we say $x = c$ is a point of local maxima if $\exists\ h > 0$ such that $f(x) \le f(c) \ \forall \ x \in (c-h, c+h)$

Local Minima:- Let $y = f(x)$ is defined $\forall x \in I$ and $c \in I$, we say $x = c$ is a point of local minima if $\exists\ h > 0$ s.t. $f(x) \ge f(c) \ \forall \ x \in (c-h, c+h)$

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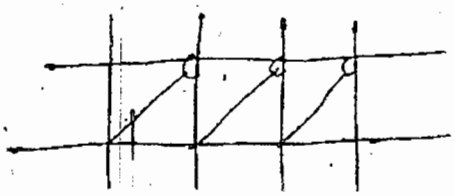
Note:- For existence of local maxima and minima, at any point, limit, continuity and differentiability is not required.

e.g.: $y = \{x\} = f(x)$



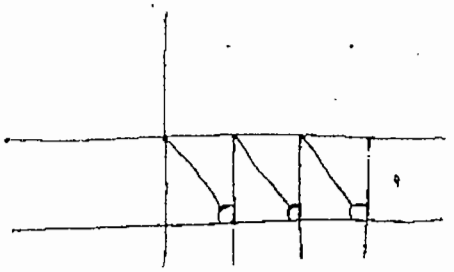
Here $f(1) = 0$

$f(1+h) > 0 \Rightarrow f(1+h) > f(1)$

and $f(1-h) > 0 \Rightarrow f(1-h) > f(1)$

Hence $x = 1, 2, 3, 4, \ldots$ are pts

$y = f(x) = -[x]$



Here $f(1) = 0$

$f(1+h) < 0 \Rightarrow f(1+h) < f(1)$

$f(1-h) < 0 \Rightarrow f(1-h) < f(1)$

Hence $x = 1, 2, 3, \ldots$ are points of local maxima.

Note:- If $f(x)$ is differentiable on I and $c \in I$ where c is a point of local extrema iff $f'(c)$ and $f'(c-\varepsilon) - f'(c+\varepsilon) < 0$ for finitely small ε.

Maxima:- If $x = c$ is a ~~point~~ stationary point.

i.e. $f'(c) = 0$.

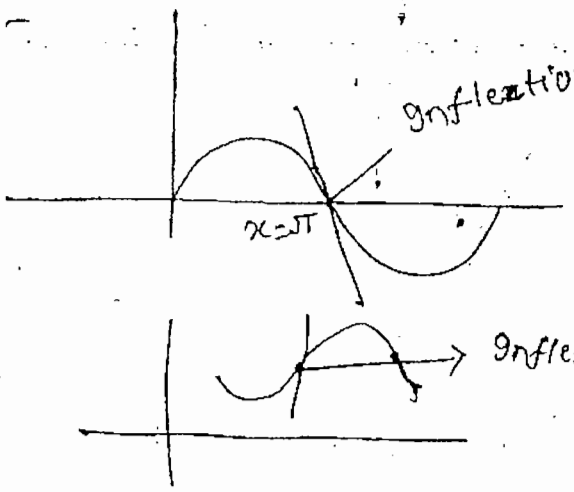
$f'(x)$ changes to its sign from +ve to -ve then $x = c$ is point of local maxima. [+, 0, −]

Minima:- and −ve to +ve then $x = c$ is point of local ~~maxima~~ minima. [−, 0, +]

Note:- 1). If $f'(x)$ does not change its sign in the nbd of c then it is called point of inflexion.

2). If at $x = c$, $f(x)$ has point of inflexion then $f'(c)$ need not be zero.

Note:-



$f(x) = \sin x \Rightarrow f'(x) = \cos x$

$f'(\pi) = \cos\pi \neq 0$

→ Point of inflection need not be stationary point.

e.g. $y = \sin x$; $x \in [\pi, \alpha]$

$y' = \cos x$

$y' = -1$ at $x = \pi$

$m = -1$

$y - 0 = -1(x - \pi)$

$y = \pi - x$

Imp

Problem:- $y = x^{1/x}$; $x \in (0, \infty)$

$\log y = \frac{1}{x} \log x \Rightarrow \frac{1}{y}\frac{dy}{dx} = -\frac{1}{x^2}\log x + \frac{1}{x^2}$

$\Rightarrow \frac{dy}{dx} = \frac{y(1-\log x)}{x^2} = 0$

$\Rightarrow \boxed{x = e}$

$f'(e) = 0$: since $\log x$ is increasing function

$f'(e-h) > 0$ and $f'(e+h) < 0$

i.e. $f'(x) > 0 \quad \forall\ x \in (0, e)$

$f'(x) < 0 \quad \forall\ x \in (e, \infty)$.

$f'(e) = 0 \Rightarrow x = e$ is a point of local maximum

Now, $x_1, x_2 \in [e, \infty)$

$x_1 < x_2$

then $f(x_1) > f(x_2)$

In particular $e, \pi \in [e, \infty)$

$f(e) > f(\pi)$

$e^{1/e} > \pi^{1/\pi}$

$\left(e^{1/e}\right)^{e\pi} > \left(\pi^{1/\pi}\right)^{e\pi}$

$\boxed{e^\pi > \pi^e}$ → Remember.

→ $\pi^2 \quad e\pi$
$x_1 \quad x_2$
$x_1 > x_2$
$f(x)\downarrow \quad x \in (e, \infty)$
$x_1 > x_2$
$f(x_1) < f(x_2)$
$\Rightarrow \boxed{(\pi^2)^{1/\pi^2} < (e\pi)^{1/e\pi}}$

→ $\sqrt{e}, \sqrt{3} < e$
$x_1 \quad x_2$
$f(\sqrt{2}) < f(\sqrt{3})$
$(\sqrt{2})^{1/\sqrt{2}} < (\sqrt{3})^{1/\sqrt{3}}$
$\Rightarrow \boxed{(\sqrt{2})^{\sqrt{3}} < (\sqrt{3})^{\sqrt{2}}}$

Application of Second Derivative :-

1). If $f'(c) = 0$, $f''(c) > 0$ then $x = c$ is a point of local minima.

2). If $f'(c) = 0$, $f''(c) < 0$ then $x = c$ is a point of local maxima.

3). If $f''(c) = 0$ but $f'''(c) \neq 0$ then $x = c$ is point of inflection.

In general $f^{n-1}(\alpha) = 0$ and $f^n(\alpha) \neq 0$

$n = 2, 3, 4, 5, 6, \cdots$; $n \neq 1$.

Case I :- n is even

max ——— $f^n(\alpha) < 0$

min ——— $f^n(\alpha) > 0$

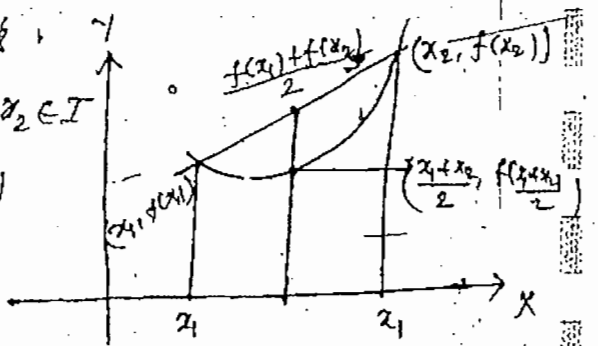
Case II :- n is odd

$x = \alpha$ is a point of inflection. —

Convex function :- A function is said to be convex on I if $\forall\ x_1, x_2 \in I$

$$f(\lambda x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

; $0 \leq \lambda \leq 1$



→ If $f(x)$ is convex in I iff $f'(x)$ is increasing $\forall\ x \in I$ i.e. $f''(x) > 0$.

→ $f(x)$ is concave iff $f''(x) < 0$.

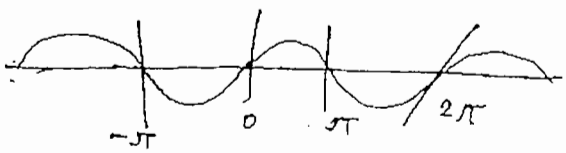
→ The point where the function changes its concavity or convexity is called point of inflection.

Imp.

Problem:- $f(x) = \sin x$, Find point of inflection.

Solution:- $f(x) = \sin x$

$f'(x) = \cos x$



$f''(x) = -\sin x = 0 \Rightarrow x = n\pi$.

↓

points of inflection.

V. Imp.

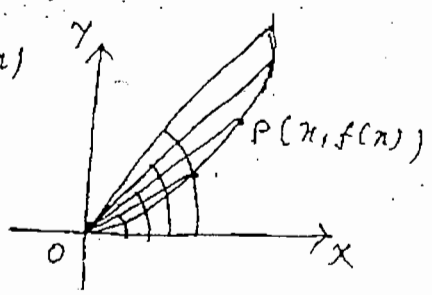
GATE.

Problem:- If $f(x)$ is defined $\forall\ x \in [0,\infty)$ and twice differentiable s.t. $f(0)=0$ & $f''(x)>0$.

further construct $g(x) = \frac{f(x)}{x}$; $x \in (0,\infty)$

Then show that $g(x)$ is increasing.

Proof:- slope of $OP = \frac{f(x)-0}{x-0} = \frac{f(x)}{x} = g(x)$



OR, Now, $g'(x) = \frac{xf'(x) - f(x)}{x^2}$;

$\phi(x) = x^2 g'(x) = xf'(x) - f(x)$

$\phi'(x) = xf''(x)$

$\phi'(x) > 0 \quad \forall\ x \in (0,\infty)$

$\phi(x)$ is increasing

$\phi(0) = \lim_{x \to 0} x^2 g'(x) = 0$

$\phi(0) = 0 \Rightarrow \phi(x) > 0 \quad ; x \in (0,\infty)$

$x^2 g'(x) > 0 \Rightarrow g'(x) > 0$

$\Rightarrow g(x)$ is increasing.

Problem:- Given $0 \leq f'(x) \leq f(x)$; $x \in [0, \infty)$ & if $f(0)=0$, then f is identically zero.

Solution:- $g(x) = e^{-x} f(x) \geq 0$ ——(i)

$$g'(x) = e^{-x}[f'(x) - f(x)] \leq 0$$

$g(x)$ is decreasing.

$g(0) = 0 \Rightarrow g(x) \leq 0 \quad \forall x \in [0, \infty)$ ——(ii)

Now, (i), & (ii) we have

$$g(x) = 0$$

$$\Rightarrow e^{-x} f(x) = 0.$$

$$\Rightarrow \boxed{f(x) \equiv 0}$$ Hence proved.

Problem:- Given $0 \leq f'(x) \leq 2f(x)$ and $f(0)=0$ $\forall x \in [0,1]$ then $f(x) \equiv 0$.

Solution:- Construct $g(x) = e^{-2x} f(x)$

$$g'(x) = f'(x) e^{-2x} - 2f(x) e^{-2x}$$

$$\Rightarrow g'(x) = e^{-2x}[f'(x) - 2f(x)]$$

$$\Rightarrow g'(x) \leq 0$$

$$\Rightarrow g(x) \downarrow$$

$$g(0) = 0$$

$$g(x) \leq g(0) \quad \forall x \in [0,1]$$

$$g(x) = f(x) \cdot e^{-2x}$$

$$g(x) \geq 0$$

$$g(x) = 0$$

$$\Rightarrow f(x) \cdot e^{-2x} = 0$$

$$\Rightarrow \boxed{f(x) \equiv 0}$$ Hence proved

Subjective.

Problem:- If $f(x) = f(mx)$, Then show that f is constant.

Proof:- Let $x = a$.

$$f(a) = f(ma) = f(m^2a) = \cdots = f(m^n a)$$

$$x_n = m^{n-1}a \; ; \; 0 < m < 1$$

$$\langle x_n \rangle \longrightarrow 0$$

$$\langle f(x_n) \rangle \longrightarrow f(0).$$

$$f(x_n) = f(a) \quad \forall \; n \in \mathbb{N}$$

$$\langle f(x_n) \rangle \longrightarrow f(a) = f(0)$$

$$f(x) = f(0) \quad \forall \; x \in \mathbb{N}$$

$\therefore$ f is constant.

$$f(mx) = f(x) \quad \forall \; x \in \mathbb{R}$$

$$\Rightarrow f(x) = f\left(\tfrac{1}{m}x\right) \quad \forall \; x \in \mathbb{R}$$

$$\Rightarrow f(x) = f\left(\tfrac{1}{m}x\right) = f\left(\tfrac{1}{m^2}x\right) = \cdots$$

$$\overset{n \to \infty}{\Rightarrow} f(x) = f(0) \quad \forall \; x \in \mathbb{R}$$

$\therefore$ f is constant.

Subjective.

Problem:- If $f(x) = f(2x) \quad \forall \; x \in \mathbb{R}$, Then show that f is constant.

Proof:- $f(x) = f\left(\tfrac{1}{2}x\right) = f\left(\tfrac{1}{4}x\right) = \cdots$

$$f(x) = f\left(\tfrac{1}{2^n}x\right)$$

$\therefore$ $f(x) = f(0) \quad \forall \; x \in \mathbb{R}$

$\therefore$ f is constant.

Subjective

Problem:- If $f(x) = f(2x+1)$ $\forall x \in \mathbb{R}$. Then show that f is constant.

Proof:- $f(x) = f\left(\frac{x+1}{2} - 1\right)$

$\Rightarrow f(x) = f\left(\frac{x+1}{2} - 1\right) = f\left(\frac{x+1}{2^2} - 1\right) = f\left(\frac{x+1}{2^3} - 1\right) = \cdots$

$f(x) = f\left(\frac{x+1}{2^n} - 1\right)$.

$x_n = \frac{a+1}{2^{n-1}} - 1$

$x_1 = a$

$x_2 = \frac{a+1}{2} - 1$

$\langle x_n \rangle \longrightarrow -1$

$\langle f(x_n) \rangle \longrightarrow f(-1)$

$\langle f(x_n) \rangle \longrightarrow f(a)$

$f(a) = f(-1)$

$|f(x_1) - f(x_2)| = \left|f\left(\frac{x_1+1}{2^n} - 1\right) - f\left(\frac{x_2+1}{2^n} - 1\right)\right| < \varepsilon$

$\therefore$ f is constant.

Subjective

Problem:- If $f(x^2) = f(x)$ $\forall$ $x \in [0,1]$ then f is constant.

Proof:- $f(x) = f(0)$

$f(x) = f(1)$

$\lim_{h \to 0} f(1-h) = f(0) = f(1)$

$\therefore$ f is constant.

$f(a) = f(a^2) = f(a^4) = \cdots$

$f(a) = f(a^2) = f(a^{2^n}) = \cdots$

$0 < a < 1$

$f(a) = f(0)$

$f(a) = f(a^{1/2}) = f(a^{1/4}) = \cdots$

$f(a) = f(1)$.

Properties of derivatives :-

Darboux Property (as I.V.P) :- If f is differentiable on $[a,b]$ and $f'(a) \le k \le f'(b)$ (let $f'(a) < f'(b)$) then $\exists\ c \in (a,b)$ s.t. $\boxed{f'(c) = k}$.

i.e. $f'(x)$ attains every value between $f'(a)$ and $f'(b)$

i.e. the image of an interval under $f'(x)$ is either interval or singleton. (constant)

i.e. if f is differentiable on I, $f'(x)$ satisfies I.V.P on I.

→ Function may not be continuous even when I.V.P.

V.Imp. → If $f'(x)$ belongs to A $\forall\ x \in I$ where A can not contain interval as subset then $f(x)$ is linear.

$$f'(I) \subseteq \mathbb{N}$$

$$f'(x) = c \quad \forall\ x \in I$$

$$\boxed{f(x) = cx + d}$$

Primitive :- If $F'(x) = f(x)$ then F is called a primitive of f.

→ Primitive is not unique. f need not to be continuous.

V.Imp → Even discontinuous function may have primitive.

Problem :- $f(x) = \begin{cases} -1 & ; -1 < x \le 0 \\ 1 & ; 0 < x \le 1 \end{cases}$ discontinuous at $x = 0$

Find $F(x)$ s.t. $F'(x) = f(x)$

Not find s.t. $F(x)$ because $f(x)$ is not satisfied I.V.P.

Note:- As $f(x)$ satisfied I.V.P. we can say a function which fails to satisfy I.V.P. can not have primitive.

9mp

Note:- If $f'(x)$ is monotonic then $f'(x)$ is continuous.

eg:- $$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x \sin 1/x - \cos\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$f'(x)$ is not continuous at $x=0$.
When it has IInd kind discontinuity.

V.V.9mp

Formula:- $$f(x) = \begin{cases} x^n \sin(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \quad n \in N$$

(i). If $n=1 \Rightarrow f(x)$ is continuous but not differentiable.

(ii) If $n=2 \Rightarrow f(x)$ is differentiable but $f'(x)$ is not continuous.

(iii) If $n>2 \Rightarrow f^{\left[\frac{n}{2}\right]}(x)$ exist and $f^{\left[\frac{n}{2}\right]}x$ is continuous if n is odd
and discontinuous if n is even.

eg:- $$f(x) = \begin{cases} x^{1010} \sin(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$f^{1005}(x)$ exist and $f^{1005}(x)$ is not continuous.

Imp.

Problem:- If $f(x)$ is defined $\forall\ x \in [1,\infty)$

(i) $f'(x) = \dfrac{1}{x^2 + [f(x)]^2}$

(ii) $f(1) = 1$

Show that $\lim\limits_{x\to\infty} f(x)$ exists $< 1 + \frac{\pi}{4}$.

Solution:- Given $f'(x) > 0$

$\Rightarrow f(x)$ is increasing function

Key ← Now $f(x) = \int_1^x f'(t)\,dt + f(1)$

$$f(x) = \int_1^x \frac{dt}{t^2 + (f(t))^2} + 1$$

$$f(x) < \int_1^x \frac{dt}{1+t^2} + 1 \qquad ; f(1) = 1 \quad f(x) \geq 1 \quad \forall x \in (1,\infty)$$

$$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} [\tan^{-1} t]_1^x + 1$$

i.e. $$\lim_{x\to\infty} f(x) < \lim_{x\to\infty} \left[\tan^{-1} x - \frac{\pi}{4}\right] + 1 = \frac{\pi}{2} - \frac{\pi}{4} + 1$$

Hence $\boxed{\lim\limits_{x\to\infty} f(x) < \frac{\pi}{4} + 1}$ Hence proved.

Note:- If $f(x)$ is non-negative differentiable function defined on $\mathbb{R}$ then if $f(\alpha) = 0 \Rightarrow \alpha$ is repeated roots of $f(x)$ i.e. $f'(\alpha) = 0$.

Means of repeated root :- If $f(\alpha) = f'(\alpha) = f''(\alpha) = 0$ and $f'''(\alpha) \neq 0$, then we say α is multiple root of order 3.

Imp → Every root of $f(x)$ is also root of $f'(x)$ if $f(x) \geq 0$.

→ If $f'(a) \cdot f'(b) < 0$ then $\exists\ c \in (a,b)$ s.t. $f'(c) = 0$

Proof :- Construct $\phi(x) = f(x) - kx$

$$\Rightarrow \phi'(x) = f'(a) - k$$

$$\phi'(a) < 0$$

$$\phi'(b) < 0$$

Some Important Theorems

Rolle's Theorem:- If f is

(i) continuous on $[a,b]$

(ii) differentiable on (a,b)

(iii) $f(a) = f(b)$

then $\exists\ c \in (a,b)$ s.t. $f'(c) = 0$.

Note:- (i) c need not be unique.

(ii) If a function $f(x)$ satisfied the condition of Rolle's theorem then $\exists$ a tangent parallel to x-axis.

e.g:- $f(x) = \sin x \ ; x \in [0, 6\pi]$.

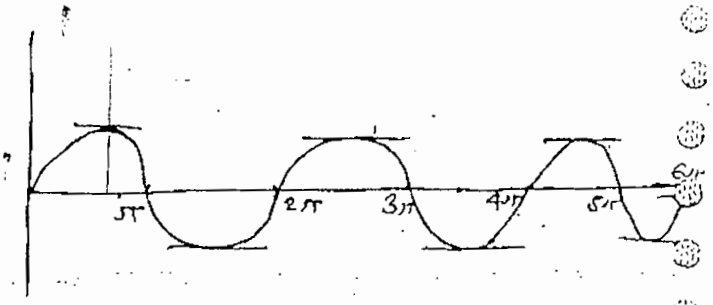
$f(0) = f(6\pi) = 0$

$f'(x) = \cos x$

$f'(c) = 0$

$\Rightarrow \cos c = 0$

$\Rightarrow c = (2n+1)\frac{\pi}{2}$



6 - c exist.

Imp.

OR, i.e between any two roots of $f(x)$ there lies the roots of $f'(x) = 0$.

Imp

Problem:- If $f(x)$ is thrice differentiable on $[0,1]$ and $f(x)$ is non-negative and, $\exists\ \alpha_1 \neq \alpha_2 \in (0,1)$ s.t. $f(\alpha_1) = f(\alpha_2) = 0$

Then show that $\exists\ c \in (0,1)$ s.t. $f'''(c) = 0$.

Solution:- Now $f(\alpha_1) = f(\alpha_2) = 0$

$c \in (\alpha_1, \alpha_2)$

$f'(c_1) = 0$

$\alpha_1 \qquad c_1 \qquad \alpha_2$

$[\alpha_1, c_1] \qquad (c_1, \alpha_2]$

$f'(\alpha_1) = 0 = f'(c_1) \qquad f'(c_1) = f'(\alpha_2) = 0$

$f''(c_2) = 0 \qquad f''(c_3) = 0$

$f'''(c) = 0 \quad ; \quad c \in (0,1)$

Problem:- If $f(x) = x^3 - 3x + \lambda \; ; \; x \in (0,1)$.

For what value of λ, $f(x)$ has two distinct roots in $(0,1)$.

Solution:- Two roots $\alpha, \beta \in (0,1)$

$$f(\alpha) = 0 = f(\beta)$$

By Rolle's theorem on $f(x)$ in (α, β)

$\Rightarrow c \in (\alpha, \beta)$

s.t. $f'(c) = 0 \Rightarrow 3c^2 - 3 = 0$

$\Rightarrow c = \pm 1 \in (\alpha, \beta) \subset (0,1)$

which is not possible

Hence no value of λ, $f(x)$ has two distinct roots in $(0,1)$.

$3c^2 - 3 = 0$

$\Rightarrow c^2 - 1 = 0$

$\Rightarrow \boxed{c = \pm 1}$

V. Imp.

Problem:- $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

s.t. $a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} = 0$

then show that $\exists\ c \in (0,1)$ s.t. $\boxed{f(c)=0}$

Solution:- Let $\phi(x) = \int f(x)\,dx = a_0 x + \frac{a_1 x^2}{2} + \frac{a_2 x^3}{3} + \cdots + \frac{a_n x^{n+1}}{n+1} + C$

$\phi(0) = C$, $\phi(1) = a_0 + \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} + C = C$

i.e. $\phi(0) = \phi(1)$

and $\phi(x)$ is continuous on $[0,1]$.

$\phi(x)$ is differentiable in $(0,1)$

then $\exists\ c \in (0,1)$ s.t. $\phi'(c) = 0$

$\Rightarrow \boxed{f(c)=0}$ Hence proved

$\longrightarrow$ $f(x) = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $\quad c \in [0, 1/\pi]$.

There are infinite points where $f'(c)=0$

$A = \{ c \in [0, 1/\pi] : f'(c) = 0 \}$

then A is countably infinite.

$\longrightarrow$ $f(x) = \begin{cases} x \sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $\longrightarrow$ not differentiable at $x=0$

$c \in [0, 1/\pi]$.

$A = \{ c \in (0, \frac{1}{\pi}) : f'(c) = 0 \}$ i.e. $\tan(1/c) = \frac{1}{c} : c \in (0, \frac{1}{\pi})$

then A is countably infinite.

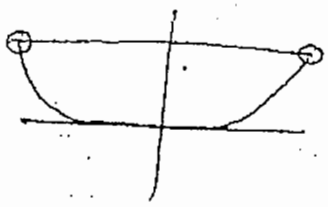
→ $f(x)=\alpha$; $\alpha \in \mathbb{R}$ (constant function)

$A = \{ c \in [a,b] : f'(c)=0 \}$

then A is uncountable.

→ $y = 2(x+1)^2$

$$y = \begin{cases} 2(x+1)^2 & ; x \le -1 \\ 0 & ; -1 \le x \le 0 \\ 2x^2 & ; x \ge 0 \end{cases}$$



Function is not constant even if function is uncountable.

Result:- (i) $F''(x) = e^x F(x)$ $\forall x \in [0,1]$

(ii) $F(0) = F(1) = 0$

Then F is identically zero or F is constant.

Result:- If f is differentiable $\forall x \in [0,1]$

$f: [0,1] \longrightarrow [0,1]$

s.t. $|f'(x)| \le \frac{1}{2}$

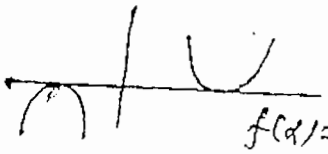
then $\exists$ unique $c \in [0,1]$ s.t. $\boxed{f(c)=c}$.

→ Between two zeros of a differentiable function there lies a zeros of its derivative.

→ There is a line parallel to x-axis and tangent the curve then the function of derivative is zero at that point.

→ If a function is differentiable and does not change its sign throughout interval then every zeros of f(x)

it also a zero of $f'(x)$.



→ $f(x)$, no attains +ve maxima, −ve minima then function is identically zero.

→ $F(0) = F(1) = 0$

Let $F(x)$ is non-constant

$\exists\ c \in (0,1)$.

s.t. $\boxed{f(c) \neq 0}$

→ If $f(x)$ is differentiable on $[a,b]$ and $f(a)=f(b)=0$ then $f(x)$ attains +ve maxima or −ve minima or both.

Problem:- If $f(x)$ is n times differentiable on $[0,1]$ s.t. $f(0) = f'(0) = \cdots = f^{n-1}(0) = f(1) = 0$ then show that $\exists\ c \in [0,1)$

s.t. $\boxed{f^n(c) = 0}$

Solution:- Here $f(0) = f(1) = 0$

then $\exists\ c_1 \in (0,1)$ s.t. $f'(c_1) = 0$

Now, again

$$f'(0) = f'(c_1) = 0$$

then $\exists\ c_2 \in (0, c_2) \subset (0,1)$

s.t. $f''(c_2) = 0$

Similarly

$$f^{n-1}(0) = f^{n-1}(c_{n-1}) = 0$$

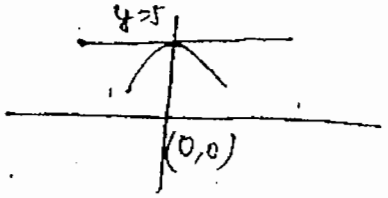
then $\exists\ c \in (0, c_{n-1}) \subset [0,1)$

s.t. $\boxed{f^n(c) = 0}$ Hence proved

NET.

Problem :- If $f(x)$ is differentiable on $\mathbb{R}$

$$f(0)=5, \quad f(x)<5 \quad \forall x \in \mathbb{R}-\{0\}$$



(i) $f(x)$ is bounded

(ii) $f(x)$ is monotonic

(iii) $f'(0)=0$

(iv) $f(x)$ is neither bdd above nor bdd below.

V.Imp

Result :- Let $f(x)$ be defined on $[0,1]$ and satisfying the condition $f''(x)=e^x f(x)$, then $f(x)$ cannot have +ve maxima and cannot have −ve minima. Further if $f(0)=f(1)=0$ then $f(x)\equiv 0$.

Proof :- Let c be the only point s.t. $f'(c)=0$

Case I : Let $f'(c-h)>0$ and $f'(c+h)<0$.

$\Rightarrow$ Function is strictly increasing in $(0, c-h)$.

Let there is a point $t \in (0,1)$ s.t. $f(t)\neq 0$ & $f(t)>0$.

Let $S=\{f(x)\in[0,1] : f(x)>0\}$ & S is non-empty.

Now, $S\neq\phi$

Since $f(x)$ is continuous on $[0,1]$

$\Rightarrow$ $f(x)$ is bounded

$\Rightarrow$ $S \subseteq$ Range f

i.e. $S \subseteq f[0,1]$

$\Rightarrow$ S is also bounded.

Then $\sup S$ exists

Let $\sup S=\lambda$

Now as f is continuous on $[0,1]$ $\exists\ \alpha \in [0,1]$ s.t.

$$f(\alpha)=\lambda$$

$\Rightarrow$ $f(x)$ has a +ve maxima which is not possible

Similarly the other case Hence proved

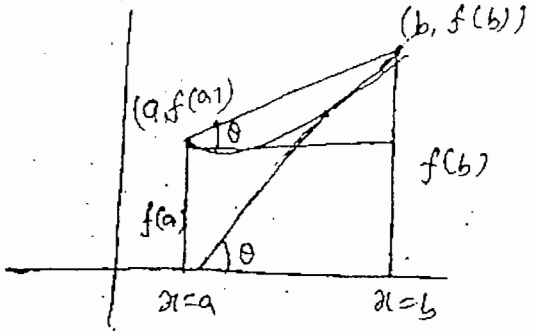
Lagrange's Mean Value Theorem (LMVT):- If $f(x)$ is

(i) continuous on $[a,b]$

(ii) differentiable on (a,b)

then $\exists\ c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



i.e. To any chord of a smooth curve there is a tangent parallel to that chord.

Proof:- $\phi(x) = f(x) + \lambda x$

(continuous & diff)

$\Rightarrow$ $\phi(x)$ is continuous on $[a,b]$ and differentiable on (a,b)

Choose λ such that $\phi(a) = \phi(b)$

$$\Rightarrow f(a) + \lambda a = f(b) + \lambda b$$

$$\Rightarrow \lambda = -\frac{f(b)-f(a)}{b-a}$$

Then with this λ we can say $c \in (a,b)$ s.t. $\phi'(c) = 0$

$$\Rightarrow f'(c) + \lambda = 0$$

$$\Rightarrow f'(c) = -\lambda$$

$$\Rightarrow \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}}$$ Hence proved.

Note:- (i) The c obtained in LMVT need not to be unique (may be finite or infinite, countable or uncountable).

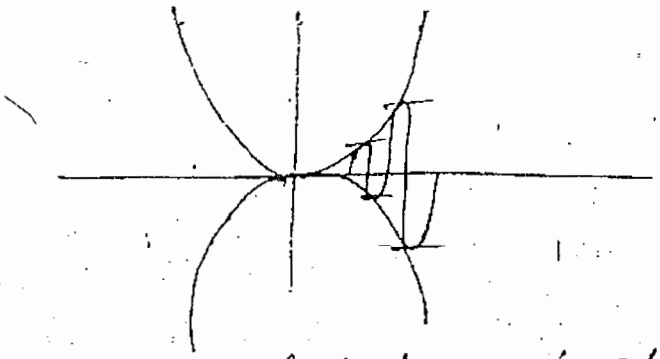
(ii) The collection of such c is non-empty.

(iii) LMVT is obtained by Rolle's theorem.

e.g. :- In case of line, $f(a) = f(b) = f(c) = 0$.
Set is whole interval of $[a, b]$.

e.g. :- $f(x) = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

countably infinite set



The collection of such c is always non-empty.
The set can be uncountable.

→ $c \in (a, b)$ then $c = a + \theta h$ where $0 < \theta < 1$
and $b - a = h$

Then $F(b) - F(a) = (b-a) F'(c)$.

$$\boxed{F(a+h) = F(a) + h F'(a+\theta h)} \rightarrow \text{L.M.V.T.}$$

After expansion

$$F(a+h) = F(a) + h F'(a) + \frac{h^2}{2!} F''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} F^{n-1}(a) + R_n$$

Where $R_n = \frac{h^n}{n!} F^n(a+\theta h)$

→ $F(a+h) = F(a) + h F'(a) + \cdots + \frac{h^n}{n!} F^n(a+\theta h)$

(i) $F(x)$ is differentiable n times & $F^n(x)$ is continuous.

(ii) Then number of terms in expansion is $(n+1)$.

e.g. :- $f(x) = \begin{cases} x^5 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Here $f''(x)$ exist and continuous.

So, number of terms in expansion is 3.

$$f'(x) = \begin{cases} 5x^4 \sin\frac{1}{x} - x^3 \cos\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$$f''(x) = \begin{cases} 20x^3 \sin\frac{1}{x} - 5x^2 \cos\frac{1}{x} - 3x^2 \cos\frac{1}{x} - x \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

But on taking 3rd derivative we get terms $\sin\frac{1}{x}$, $\cos\frac{1}{x}$ which are not continuous at $x = 0$

e.g. :- If $f(x) = \begin{cases} x^4 \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Here $f''(x)$ exist but not continuous.

So, number of terms in expansion is 2.

Remember :- $f(x) = \begin{cases} x^n \sin\frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

$f^{\left[\frac{n}{2}\right]}(x)$ exists

If n = even, exist but not continuous

If n = odd, exist and continuous.

→ If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then $A = \{c \in (a,b) : \frac{f(b)-f(a)}{b-a} = f'(c)\}$

Then A is non-empty.

Note:- 1). If $f : [a,b] \xrightarrow{\text{continuous}} \mathbb{R}$

then $f[a,b] \subseteq [f(a), f(b)]$ (By E.V.P)

2). If $\sup f(x) = f(b)$

and $\inf f(x) = f(a)$

then $f[a,b] = [f(a), f(b)]$

3). If $f(x)$ is continuous on (a,b)

$\sup f(x) = f(k_2)$

$\inf f(x) = f(k_1)$

then $f[a,b] = [f(k_1), f(k_2)]$

→ If f is continuous and monotonic on $[a,b]$

then $k_1 = a$, $k_2 = b$

then $f[a,b] = [f(a), f(b)]$.

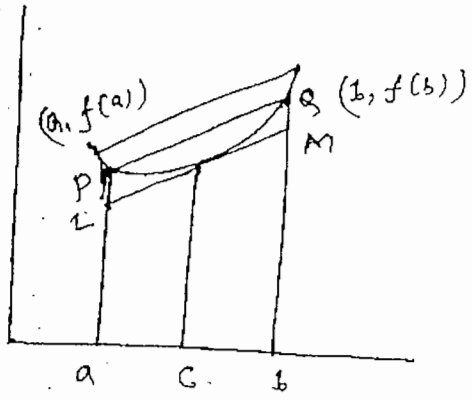
IInd Form LMVT :-

Chord of $PQ = \frac{f(b)-f(a)}{b-a}$

and slope of $LM = f'(c)$

Now, $PQ \parallel LM$

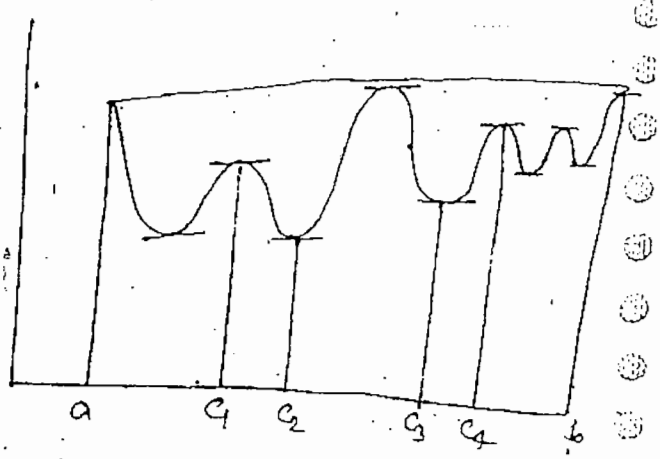
then $f'(c) = \frac{f(b)-f(a)}{b-a}$



LMVT says that a differentiable function there is a tangent parallel to any chord.

Here c is not unique.

Now, f(x) is convex / concave in [a, b] then c is obtained and is unique.



Problem :- If $f(x)$ is twice differentiable in (a,b)

$f''(x) > 0 \quad \forall \; x \in (a,b)$

$$A = \left\{ c \in (a,b) : f'(c) = \frac{f(b)-f(a)}{b-a} \right\}$$

then

(i) A is empty

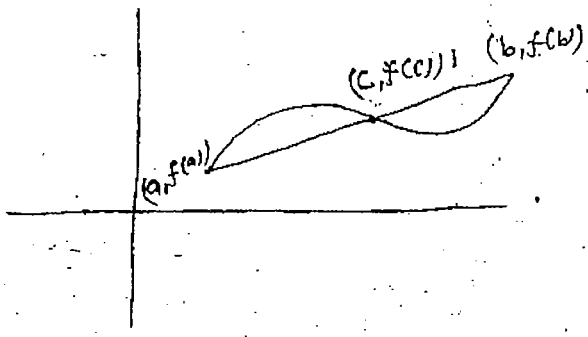
(ii) A is finite need not to be singleton.

(iii) A is singleton

(iv) None of these.

Problem:- If f is continuous on $[a,b]$ and twice differentiable on (a,b). If line segment joining $(a,f(a))$ and $(b,f(b))$ intersect the curves into third points then show that $\exists\ c \in (a,b)$

s.t. $\boxed{f''(c)=0}$



Proof:- $\exists\ c_1 \in (a,c)$

s.t. $f'(c_1) = \dfrac{f(c)-f(a)}{c-a}$

and $\exists\ c_2 \in (c,b)$

s.t. $f'(c_2) = \dfrac{f(b)-f(c)}{b-c}$

$\Rightarrow f'(c_1) = f'(c_2)$

$\Rightarrow \dfrac{f(c)-f(a)}{c-a} = \dfrac{f(b)-f(c)}{b-c}$

Hence by Rolle's theorem $\exists\ c \in (c_1,c_2)$

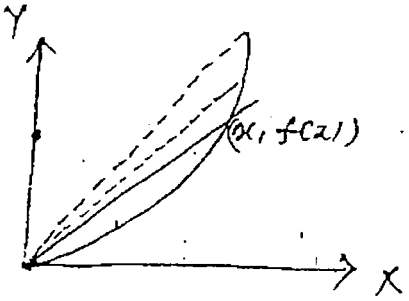
s.t. $\boxed{f''(c)=0}$

Problem:- Use L.M.V.T. to show $g(x)=\dfrac{f(x)}{x}$ is ↑ function where $f''(x)>0$ and $f(0)=0$.

Solution:- If $f''(x)>0 \rightarrow$ convex function ($\because$ chord above then convex)

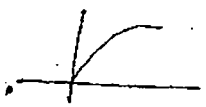
$f(0)=0 \rightarrow$ passes through 0

Slope of chord (passes through origin) $= \dfrac{f(x)-0}{x-0}$



which is increasing

Hence $g(x)$ is ↑



$f''(x)>0 \Rightarrow f'(x)$ is ↑.

Note:- A differentiable function $f(x)$ is convex iff $f'(x)$ is ↑ or, $f''(x) > 0$ provided 2nd derivative exist

→ $g(x) = \frac{f(x)}{x}$; $f''(x) > 0$

$f(0) = 0$

IInd method $f'(x)$ is ↑ $(\because f''(x) > 0)$ → Remember

We can write

$$f(x) = \int_0^x f'(t)\,dt \le \int_0^x f'(t)\,dt \quad [\because f'(x) \text{ is } \uparrow]$$

$\Rightarrow f(x) \le f'(x) \cdot x$

Now, $g'(x) = \frac{xf'(x) - f(x)}{x^2}$ $\left[\because g(x) = \frac{f(x)}{x}\right]$

$= +ve$

i.e $g'(x) > 0$

$\Rightarrow g(x)$ is increasing function.

Applications of LMVT:-

1). **Madvan Mean Value Theorem**:- If $f(x)$ is continuous on $[a,b]$ and differentiable on (a,b) then $\exists\ \lambda \in (x_1, x_2) \subset (a,b)$

s.t. $\boxed{\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(\lambda)}$

→ Let $b - a = h \Rightarrow b = a + h$

Then LMVT takes the form

$$\frac{f(a+h) - f(a)}{a+h-a} = f'(c) \quad ; \quad c \in (a, a+h)$$

or, $\boxed{f(a+h) = f(a) + hf'(a+\theta h)}$ Where $0 < \theta < 1$

↳ Generalised LMVT

And again $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a+\theta h)$; $0<\theta<1$

$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a+\theta h)$; $0<\theta<1$

Taylor's Extension :- Taylor's expansion of f near 'a'

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \cdots + \frac{h^n}{n!}f^n(a+\theta h)$$

↑ Remainder term or R_n $(0<\theta<1)$

Maclaurian's Extension :- Put $h=x$, $a=0$ in above.

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \cdots + \frac{x^n}{n!}f^n(\theta x)$$

$E(\text{error}) < 0.5 \times 10^{-r}$ Where $0<\theta<1$

Note :- If f is function of class C^∞ then

$$f(a+h) = f(a) + hf'(a) + \cdots + \frac{h^n}{n!}f^n(a) + \cdots$$

Where C^∞ = continuously differentiable.

Note :- Extensions are valid upto that derivative which is continuous as well.

eg :- $f(x) = \begin{cases} x^{12}\sin(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $x \in [0,1]$.

Extension of $f(x)$ nbd of 0 upto 5^{th} derivative because 5^{th} derivative is continuous.

Note :- When function is continuously differentiable then extension becomes series.

$\rightarrow$ $f^{\left[\frac{n}{2}\right]}$ $\begin{cases} n \text{ is odd} \\ \frac{n-1}{2} \quad n \text{ is even} \end{cases}$

$$f(x) = \begin{cases} x^n \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \quad n \in \mathbb{N}$$

$f^{\left[\frac{n}{2}\right]}$ is continuous if n is odd and discontinuous if n is even.

$\rightarrow$ $$f(x) = \begin{cases} x^{51} \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$f^{\left[\frac{51}{2}\right]}$ is continuous.

$\rightarrow$ $$f(x) = \begin{cases} x^{50} \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$f^{\left[\frac{50}{2}\right]}$ is discontinuous

U.P.S.C

Problem:- If $f(x) = (x-2)^{5/2}$

If x is expanded near '2' then evaluate @.

Solution:- $f'(x) = \frac{5}{2}(x-2)^{3/2}$

$$f''(x) = \frac{15}{4}(x-2)^{1/2}$$

$$f''(2+\theta h) = \frac{15}{4}(\theta h)^{1/2}$$

$f(2) = 0$
$f'(2) = 0$
$f''(2) = 0$

Now, by Taylor's extension

$$f(2+h) = f(2) + hf'(2) + \frac{h^2}{2!} f''(2+\theta h)$$

$$\Rightarrow h^{5/2} = 0 + 0 + \frac{h^2}{2} \times \frac{15}{4} (\theta h)^{1/2}$$

$\Rightarrow \frac{8}{15} = \theta^{1/2}$

$\Rightarrow \boxed{\theta = \frac{64}{225}}$ where $0<\theta<1$

Problem:- If $f(x) = (x-2)^{7/2}$ is expanded near '2' then evaluate θ.

Solution:- $f'(x) = \frac{7}{2}(x-2)^{5/2}$

$f''(x) = \frac{35}{4}(x-2)^{3/2}$

$f'''(x) = \frac{105}{8}(x-2)^{1/2} \Rightarrow f'''(2+\theta h) = \frac{105}{8}(\theta h)^{1/2}$

$f(2)=0$
$f'(2)=0$
$f''(2)=0$
$f'''(2)=0$

Now, by Taylor's extension.

$$f(2+h) = f(2) + hf'(2) + \frac{h^2}{2!}f''(2) + \frac{h^3}{3!}f'''(2+\theta h)$$

$$h^{7/2} = \frac{h^3}{3!}\frac{105}{8}(\theta h)^{1/2}$$

$\Rightarrow \theta^{1/2} = \frac{105}{3\times 2\times 8}$ (35)

$\Rightarrow \theta = \left(\frac{35}{16}\right)^{-2}$

$\Rightarrow \boxed{\theta = \frac{256}{1225}}$ where $0<\theta<1$

Problem :- If f is twice differentiable on $\mathbb{R}$ and

$|f''(x)| \leq M_1$, $|f(x)| \leq M_2 \quad \forall x \in \mathbb{R}$

Find an upper bound of $f'(x)$.

Show that $|f'(x)|^2 \leq 4 M_1 M_2$!

Solution :- $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x+\theta h)$

$\Rightarrow hf'(x) = f(x+h) - f(x) - \frac{h^2}{2} f''(x+\theta h)$

$\Rightarrow f'(x) = \frac{1}{h} \left\{ f(x+h) - f(x) - \frac{h^2}{2} f''(x+\theta h) \right\}$

$\Rightarrow |f'(x)| \leq \frac{1}{|h|} \left\{ |f(x+h)| + |-f(x)| + \left|-\frac{h^2}{2} f''(x+\theta h)\right| \right\}$

$\leq \frac{1}{|h|} \cdot 2M_2 + \frac{h}{2} M_1 = \phi(h)$

$\phi(h) = \frac{2M_2}{h} + \frac{h}{2} M_1$

$\phi'(h)$

Cauchy Mean Value Theorem (C.M.V.T) :- If f and g are continuous on $[a,b]$ and differentiable on (a,b) then $\exists\ c \in (a,b)$

$$\text{s.t.}\quad \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \quad ; \text{ provided } g'(c)\neq 0 \ \&\ g(b)\neq g(a)$$

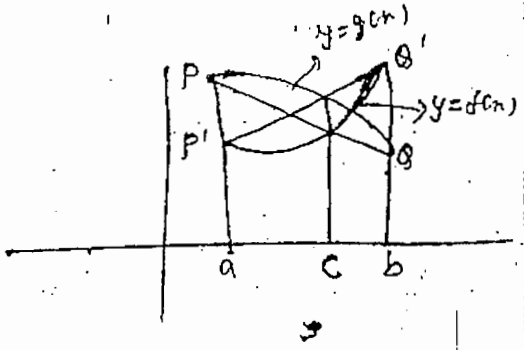
Note:- (i) The c need not be unique.

(ii) c need not be the point of intersection of curves. It just connects two functions.

Proof:- Let $\phi(x) = f(x) + \lambda g(x)$

Choose λ s.t $\phi(a) = \phi(b)$

then $\lambda = -\left[\dfrac{f(b)-f(a)}{g(b)-g(a)}\right]$ —(i)



By Rolle's theorem

then $\exists\ c \in (a,b)$

s.t. $\phi'(c) = 0$

$$f'(c) + \lambda g'(c) = 0$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = -\lambda \quad \text{—(ii)}$$

From (i) and (ii) we have

$$\boxed{\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}}$$; provided $g'(c)\neq 0$ & $g(b)\neq g(a)$

Hence proved

→ As a particular case of CMVT we can find LMVT & Rolle's theorem.

↓

give the bound of derivative.

→ (i) Rolle's theorem $\Leftrightarrow$ CMVT

(ii) CMVT $\Rightarrow$ LMVT but LMVT $\not\Rightarrow$ CMVT.

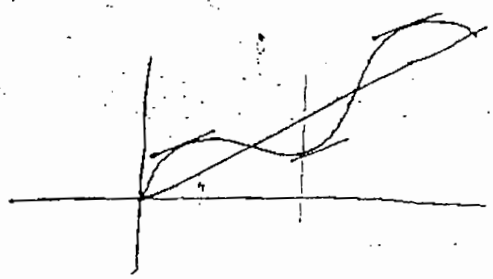
Problem:- If $A = \left\{ c \in (a,b) : f'(c) = \frac{f(b)-f(a)}{b-a} \right\}$

then A is

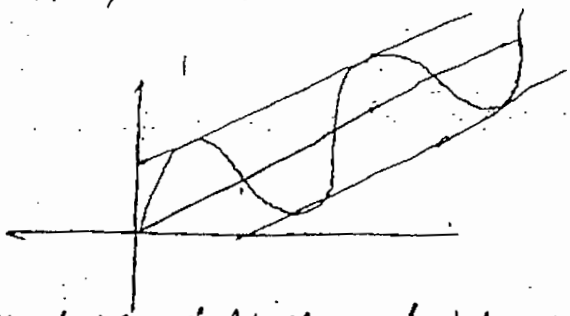
(i) finite (ii) countably infinite

(iii) infinite ~~(iv)~~ non-empty

→ $f(x) = x + \sin x \; ; \; x \in [0, 100\pi]$.



→ $f(x) = x + \sin x \; ; \; I_n = [0, n\pi]$



There are countably infinite points where $f'(c) = 0$.

Note:- 1). If $g(x) = x$; then we get LMVT

i.e. LMVT is obtained by CMVT but converse is not true.

2). If $g(x) = x$, $f(a) = f(b)$, then we get Rolle's theorem. So Rolle's theorem can be obtained by CMVT. and conversely.

Problem:- Find $c \in (a,b)$ mention in the CMVT for the following pair of functions assume the CMVT conditions are satisfied.

	$f(x)$	$g(x)$	
(i)	$\sin x$	$\cos x$	$\longrightarrow c = a+b/2$
(ii)	e^x	e^{-x}	$\longrightarrow c = a+b/2$
(iii)	$\sqrt{x}$	$1/\sqrt{x}$	$\longrightarrow c = \sqrt{ab}$
(iv)	x	$1/x$	$\longrightarrow c = \sqrt{ab}$
(v)	$1/x$	$1/x^2$	$\longrightarrow c = 2ab/a+b$

Solution:- (i) $\dfrac{f(b)-f(a)}{g(b)-g(a)} = \dfrac{f'(c)}{g'(c)}$

$$\Rightarrow \frac{\sin b - \sin a}{\cos b - \cos a} = \frac{\cos c}{-\sin c}$$

$$\Rightarrow \frac{2\cos\left(\frac{b+a}{2}\right)\cdot \sin\left(\frac{b-a}{2}\right)}{2\sin\left(\frac{b+a}{2}\right)\cdot \sin\left(\frac{a-b}{2}\right)} = -\cot c$$

$$\Rightarrow \cot\left(\frac{a+b}{2}\right) = \cot c$$

$$\Rightarrow \boxed{c = \frac{a+b}{2}}$$

(ii) $\dfrac{e^b - e^a}{e^{-b} - e^{-a}} = \dfrac{e^c}{-e^{-c}} \Rightarrow -e^{2c}$

$$\Rightarrow \frac{(e^b - e^a)(e^{a+b})}{(e^a - e^b)} = -e^{2c} \Rightarrow e^{2c} = e^{a+b} \Rightarrow \boxed{c = \frac{a+b}{2}}$$

Problem:- Show that $f(x)$ is one-one.

Here $f(x) = \cos x + \frac{1}{2}x^2 + \log x \quad ; x \in (0, \infty)$

$f'(x) = x + \frac{1}{x} - \sin x > 0 \quad x + \frac{1}{x} > 2 \quad -1 \le \sin x \le 1$

$f(x)$ is strictly increasing.

Then f is 1-1.

→ If $f(x) = 0$ fails to have roots then $f(x)$ is one-one (strictly monotonic).

Subjective

Problem:- If f is differentiable $\forall x \in [0,1]$

$$f: [0,1] \longrightarrow [0,1]$$

s.t. $|f'(x)| \le \frac{1}{2}$

then $\exists$ unique $c \in [0,1]$.

s.t $f(c) = c$.

Proof:- $g(x) = f(x) - x$

$g'(x) = f'(x) - 1$

$\Rightarrow |g'(x)| \le |f'(x) - 1|$

$\Rightarrow |g'(x)| \le |\frac{1}{2} - 1|$

$\Rightarrow |g'(x)| \le 1/2$

$g'(x)$ is strictly decreasing.

$g(x)$ can have at most one zero.

Hence $\boxed{f(c) = c}$ i.e unique fixed point.

Uniform Continuous

Definition:- Let $y=f(x)$ & $x_1 \in I$ need not be closed interval, we say f is uniformly continuous on I if for any $\varepsilon>0$, $\exists\ \delta>0$ (depends only on ε and independent of x) such that

$$|f(x_1)-f(x_2)|<\varepsilon \quad \forall\ x_1, x_2 \in I$$

s.t. $|x_1-x_2|<\delta$.

To show uniform continuity

Algorithm:-

① $|f(x_1)-f(x_2)| = |x_1-x_2|\ |\phi(x_1,x_2)|$

or, $|f(x_1)-f(x_2)| \le |x_1-x_2|\ |\phi(x_1,x_2)|$

$\le M\,|x_1-x_2| < \varepsilon \quad \forall\ x_1, x_2 \in I$

$|f(x_1)-f(x_2)| \le M\,|x_1-x_2| < \varepsilon$

Let $\frac{\varepsilon}{M} = \delta$

then $|x_1-x_2|<\delta$.

e.g.:- ① $f(x)=x$ is uniformly continuous on $\mathbb{R}$. (by graph)

Proof:- $|f(x_1)-f(x_2)| = |x_1-x_2| < \varepsilon$

Let $\delta=\varepsilon$

i.e. $|x_1-x_2|<\delta$

Thus $f(x)$ is uniformly continuous on $\mathbb{R}$.

e.g.: (ii) $f(x)=\sin x$ is uniformly continuous on $\mathbb{R}$. (by graph)

Proof:- $|f(x_1)-f(x_2)| = |\sin x_1 - \sin x_2| = \left|2\sin\frac{x_1-x_2}{2}\cos\frac{x_1+x_2}{2}\right|$

$\le 2\left|\sin\frac{x_1-x_2}{2}\right| \le 2\left|\frac{x_1-x_2}{2}\right| = |x_1-x_2| < \varepsilon=\delta$

$\Rightarrow |f(x_1)-f(x_2)|<\varepsilon$ whenever $|x_1-x_2|<\varepsilon=\delta$.

Thus $f(x)$ is uniformly continuous on $\mathbb{R}$.

e.g (iii) $f(x)=x^3$; $x\in(0,3)$ is uniformly continuous.

Proof:- $|f(x_1)-f(x_2)| = |x_1^3-x_2^3| = |x_1-x_2|\,|x_1^2+x_2^2+x_1x_2|$

$\leq |x_1-x_2|\cdot 27 < \varepsilon$

x^3 is increasing in $(0,3)$ then $M=27$

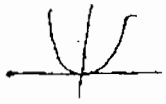
Put $\frac{\varepsilon}{27}=\delta$

i.e $|x_1-x_2|<\delta$.

Problem:- Show that $f(x)=x^2$; $x\in\mathbb{R}$ continuous throughout $\mathbb{R}$ but it is NOT uniformly continuous on $\mathbb{R}$.

Solution:- Let $x_1, x_2 \in \mathbb{R}$ then

$|f(x_1)-f(x_2)| = |x_1^2-x_2^2| = |x_1-x_2|\,\underbrace{|x_1+x_2|}_{\text{unbdd}}$



Let $x_1=\sqrt{n+1}$, $x_2=\sqrt{n}$

$x_1-x_2 = \sqrt{n+1}-\sqrt{n}$

$|x_1-x_2|<\delta$ when $n\to\infty$

$|f(x_1)-f(x_2)| = |n+1-n| = 1 > \frac{1}{2}$

Hence $f(x)=x^2$ is not uniformly continuous on $\mathbb{R}$.

Note (i) Continuity → Property of a function at a point.

(ii) Uniform continuity → Property of a function on a set.

(iii) Graph parallel to y-axis → Fails uniform continuity.

(iv) Graph parallel to x-axis → Always uniform continuity.

(v) Graph fluctuate then not uniform continuity.

Disprove to uniform continuity

Algorithm:- If for any δ we can find $x_1, x_2 \in I$ s.t. $|x_1 - x_2| < \delta$, $|f(x_1) - f(x_2)| > c$ then $f(x)$ fails to uniform continuity on I.

Problem:- ① $f(x) = \sin\frac{1}{x}$; $x \in (0, \frac{\pi}{2})$ is not U.C.

Solution:- Let $x_1 = \frac{1}{n\pi/2} < \delta$

$$x_2 = \frac{1}{(n+1)\pi/2} < \delta$$

$$|f(x_1) - f(x_2)| = 1$$

$$|x_1 - x_2| < \delta$$

V.Imp ⇒ $f(x) = \sin\frac{1}{x}$; $x \in (0,1)$ is not U.C.

V.Imp ⇒ But $f(x) = \sin\frac{1}{x}$ is U.C. on (a, ∞), $a > 0$

Problem:- ② $f(x) = \frac{1}{x}$; $x \in (0,1)$ is not U.C.

Solution:- Let $x_1 = \frac{1}{n} < \delta$, $x_2 = \frac{1}{n+1} < \delta$

$$\Rightarrow |x_1 - x_2| = \left|\frac{1}{n} - \frac{1}{n+1}\right| < \delta$$

$$|f(x_1) - f(x_2)| = \left|n - (n+1)\right| = 1 > \tfrac{1}{2} = c$$

Hence $f(x) = \frac{1}{x}$; $x \in (0,1)$ is not U.C.

Note:- $f(x) = \frac{1}{x}$ on (a, ∞), $a > 0$ then

$$|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \frac{|x_1 - x_2|}{|x_1 x_2|} \le \frac{1}{a^2}|x_1 - x_2| < \varepsilon$$

$\Rightarrow |f(x_1) - f(x_2)| < \varepsilon$ when $|x_1 - x_2| < \varepsilon a^2 = \delta$

V.Imp ⇒ $f(x) = \frac{1}{x}$ is U.C. on (a, ∞), $a > 0$

But $f(x) = \frac{1}{x}$ is not U.C. on $(0, \infty)$.

Problem:- ③ $f(x) = \sin x^2$; $x \in R$ is not U.C.

Solution:- Let $x_1 = \sqrt{(2n+1)\frac{\pi}{2}}$, $x_2 = \sqrt{(2n)\frac{\pi}{2}}$

$$|x_1 - x_2| < \delta$$

$$|f(x_1) - f(x_2)| = \left|\sin(2n+1)\frac{\pi}{2} - \sin n\pi\right|$$

$$= |1 - 0|$$

$$= 1$$

Thus $f(x) = \sin x^2$; $x \in R$ is not U.C.

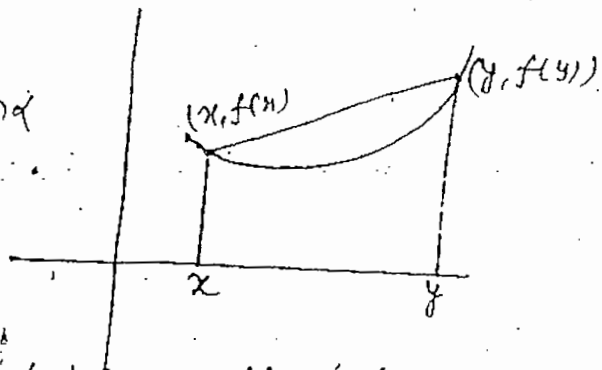
→ $f(x) = \sin x^2$ is uniform continuous in $(0, 10^{100})$.

Note:- On finite interval (a, b), $a > 0$ all above functions. (Prob ①, ②, ③) are U.C.

→ $f(x) = x^2$; $x \in (-a, a)$ is uniformly continuous.

Lipschitz function

$$\frac{f(y) - f(x)}{y - x} = \frac{f(x) - f(y)}{x - y} = \tan\alpha$$



Definition:- A function f is said to be Lipschitz if slope of the chord joining any two points on the interval is bounded.

i.e $$\frac{|f(x) - f(y)|}{|x - y|} \leq M$$

i.e. $$|f(x) - f(y)| \leq M|x - y|$$

→ If Lipschitz constant is less than 1 then the function f has unique fixed point if it exists.

Result:- (i) Lipschitz functions are uniformly continuous.

(ii) U.C. does not imply Lipschitz.

e.g:- $f(x) = \sqrt{x}$; $x \in (0, \infty)$ is U.C. but not Lipschitz.

Some Useful Test:-

(i) If f is continuous on $\mathbb{R}$ then it is uniformly continuous on every interval of finite length.

(ii) If f is continuous on I (an interval) and limit exists at the end points of the interval finitely then function is U.C.

(iii) If f is Lipschitz then it is U.C.

(iv) If f is continuous on $[a,b]$ then it is uniformly continuous on $[a,b]$.

(v) If f is differentiable on (a,b) and derivative is bounded i.e. $|f'(x)| \leq k$ then it is uniformly continuous. but not conversely. (By LMVT)

i.e If $f'(x)$ is bounded on I then f is U.C. on I but not conversely.

e.g:- $f(x) = \sqrt{x}$, $x \in (0,1)$

Derivative is not bounded but it is U.C.

Proof:- By LMVT

$$f'(c) = \frac{f(b) - f(a)}{b-a} \leq k. \quad [\because f'(x) \leq k \ \forall x \in (a,b)]$$

$$\Rightarrow |f(b) - f(a)| \leq k|b-a|$$

$\Rightarrow |f(b)-f(a)| < \varepsilon$ when $|b-a| < \frac{\varepsilon}{k} = \delta$

Thus $f(x)$ is uniformly continuous.

$\rightarrow$ $f(x) = \begin{cases} x \sin\frac{1}{x} & ; x \in (-1,1) - \{0\} \\ 0 & ; x = 0 \end{cases}$

$f(x)$ is continuous and $\lim_{h\to 0} f(-1+h)$ & $\lim_{h\to 0} f(1-h)$ both exists finitely. So $f(x)$ is U.C.

$\rightarrow$ $f(x) = x \sin\frac{1}{x}$; $x \in (0,1)$ is U.C.

Result: **Uniform continuity $\Rightarrow$ Continuity**

Warning: If f and g are uniformly continuous then $f \cdot g$ and f/g need not be uniformly continuous.

e.g (i) $f(x) = 1$; $x \in (0,1)$ is U.C.

$g(x) = x$; $x \in (0,1)$ is U.C.

But $\frac{f(x)}{g(x)} = \frac{1}{x}$; $x \in (0,1)$ is not U.C.

e.g (ii) $g(x) = x$; $x \in R$ is U.C.

$g(x) \cdot g(x) = x^2$; $x \in R$ is not U.C.

e.g (iii) $f(x) = 1$; $x \in [0,1]$ is U.C.

$g(x) = x$; $x \in [0,1]$ is U.C.

But $f(x) \cdot g(x) = x$ is U.C. on R;

$\rightarrow$ Product and ratio of U.C. functions need not be U.C.

Remarks:-

⇒ ~~If length of interval is finite then f(x) is bounded.~~ continuous function on closed & bdd interval is bdd there in.

⇒ If $f(x)$ is continuous and bounded on an interval of finite length then $f(x)$ is U.C.

⇒ Uniform continuity $\not\Rightarrow$ boundedness

e.g.:- $f(x) = x$ on $\mathbb{R}$.

⇒ Continuity + boundedness $\not\Rightarrow$ Uniform continuity.

e.g.:- $f(x) = \sin x^2$; $x \in \mathbb{R}$ is continuous, differentiable, bounded but not uniform continuous.

M.M. 9mp

Problem:- $\sum_{n=1}^{\infty} [n\{f(1/n) - f(-1/n)\} - 2f'(0)]$

Show that the series is convergent where $f(x)$ is thrice differentiable on $[-1,1]$ i.e. $f \in C^3[-1,1]$

[Hint - Meclaurian Expansion]

Solution:- By Meclaurian Expansion

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(\theta x) \quad ; 0 \le \theta < 1$$

$$\Rightarrow f(1/n) = f(0) + \frac{1}{n}f'(0) + \frac{1}{2n^2}f''(0) + \frac{1}{3!n^3}f'''(\theta/n)$$

and $$f(-1/n) = f(0) - \frac{1}{n}f'(0) + \frac{1}{2!n^2}f''(0) - \frac{1}{3!n^3}f'''(\theta/n)$$

Again $$f(1/n) - f(-1/n) = \frac{2}{n}f'(0) + \frac{2}{3!n^3}[f'''(\theta/n) - f'''(-\theta/n)]$$

or, $$n[f(1/n) - f(-1/n)] = 2f'(0) + \frac{1}{3n^2}[f'''(c_1) + f'''(c_2)]$$

$$\Rightarrow n[f(1/n) - f(-1/n)] - 2f'(0) = \frac{2M}{3n^2}$$

where $c_1 = \theta/n$, $c_2 = -\theta/n$

since $f \in C^3[-1,1]$, f is bdd i.e. $|f'''(x)| \le M$

then $\sum_{n=1}^{\infty} \frac{2M}{3n^2} = \frac{2M}{3} \sum_{n=1}^{\infty} \frac{1}{n^2}$ → convergent series.

Hence proved

M.M. 9mp

Problem :- Let $f(x)$ be a function of twice differentiable i.e. $f \in C^2$ with bounded derivatives.

Let further $A = \sup_{x \in \mathbb{R}} |f(x)|$, $B = \sup_{x \in \mathbb{R}} |f''(x)|$

Show that $\sup_{x \in \mathbb{R}} |f'(x)| \leq 2\sqrt{AB}$

[Hint: LMVT, Taylor's Expension]

Proof :- $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$; $0<\theta<1$

Now, $hf'(a) = f(a+h) - f(a) - \frac{h^2}{2!} f''(a+\theta h)$; $h>0$

$\Rightarrow f'(a) = \frac{f(a+h)}{h} - \frac{f(a)}{h} - \frac{h}{2!} f''(a+\theta h)$

$\Rightarrow |f'(a)| \leq \frac{1}{h} |f(a+h)| + \frac{1}{h} |f(a)| + \frac{h}{2} |f''(a+\theta h)|$

$\leq \frac{2A}{h} + \frac{A}{h} + \frac{h}{2} \cdot B$

$\leq \frac{2A}{h} + \frac{Bh}{2} = \phi(h)$

Now, $\phi(h) = \frac{2A}{h} + \frac{Bh}{2}$

$\phi'(h) = -\frac{2A}{h^2} + \frac{B}{2}$ Put $\phi'(h) = 0$

$\Rightarrow h = \pm 2\sqrt{A/B}$

But $h>0$ then $h = 2\sqrt{A/B}$

Hence $|f'(a)| \leq \max \phi(h)$ or $\min \phi(h)$

$\leq \frac{2A}{2}\sqrt{B/A} + \frac{B}{2} 2\sqrt{A/B} \leq 2\sqrt{AB}$

i.e. $\sup_{x \in \mathbb{R}} |f'(x)| \leq 2\sqrt{AB}$ Hence proved

Unit-5 :- Sequence and series of functions

Sequence of functions

Defⁿ :- $\langle f_n(x) \rangle = \{ f_1(x), f_2(x), \ldots, f_n(x), \ldots \}$; $x \in I$ define a sequence of functions. We can call it as family of sequence of real number and member of this family can be obtained choose different x.

i.e. Let $f_n(x)$ be defined $\forall x \in I$ & $n \in \mathbb{N}$ then the sequence $\langle f_n(x) \rangle$ is defined as sequence of functions.

Note :- For different x, e.g put $x = \alpha$, we get $\langle f_n(\alpha) \rangle$ which represents sequence of real number.

Hence we can consider that a sequence of functions is a family of sequences of real numbers.

R_f = set of all real valued functions defined on I.

$$\phi : \mathbb{N} \longrightarrow R_f$$

$$\phi(1) = f_1$$
$$\phi(2) = f_2$$
$$\vdots$$
$$\phi(n) = f_n$$

We take intersection of $f_1, f_2, \ldots f_n$.

$\langle f_1, f_2, \ldots f_n, \ldots \rangle$

Varing x we get different members of this family.

E.g :- (i) $f_n(x) = x^n$, $\langle f_n(x) \rangle = \langle x, x^2, x^3, \ldots x^n, \ldots \rangle$.

(ii) $f_n(x) = \dfrac{nx}{1+nx}$

(iii) $f_n(x) = e^{-nx}$

Limit Function :- $\lim_{n\to\infty} f_n(x) = f(x)$ then $f(x)$ is called limit function.

i.e. When $n\to\infty$ then definition of $f_n(x)$ obtained for various values of x is called the limit function of sequences.

e.g :- ① $f_n(x) = \begin{cases} n^2x & 0 \le x \le 1/n \\ -n^2x + 2n & 1/n \le x \le 2/n \\ 0 & 2/n \le x \le 1 \end{cases}$

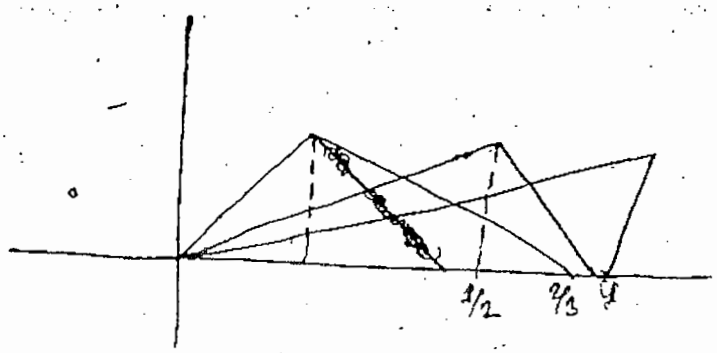
Find $f(x) = ?$

Solution :- $f_1(x) = \begin{cases} x & ; \ 0 \le x \le 1 \\ -x+2 & ; \ x = 1 \end{cases}$

$f_2(x) = \begin{cases} 4x & ; \ 0 \le x < 1/2 \\ -4x+4 & ; \ 1/2 \le x \le 1 \end{cases}$

$f_3(x) = \begin{cases} 9x & ; \ 0 \le x < 1/3 \\ -9x+6 & ; \ 1/3 \le x < 2/3 \\ 0 & ; \ 2/3 \le x \le 1. \end{cases}$

$\lim_{n\to\infty} f_n(x) = f(x) = 0 \ ; \ x \in [0,1]$.



e.g.: ① $f_n(x) = \begin{cases} -n^2x & ; \ 0 \le x < 1/2n \\ -n^2x + n & ; \ 1/2n \le x < 1/n \\ 0 & ; \ 1/n \le x \le 1 \end{cases}$

Find $f(x) = ?$

Solution:- $f_1(x) = \begin{cases} -x & ; \ 0 \le x < 1/2 \\ -x+1 & ; \ 1/2 \le x < 1 \end{cases}$

$f_2(x) = \begin{cases} -2x & ; \ 0 < x < 1/4 \\ -4x+2 & ; \ 1/4 \le x < 1/2 \\ 0 & ; \ 1/2 \le x \le 1 \end{cases}$

$f_3(x) = \begin{cases} -3x & ; \ 0 \le x < 1/6 \\ -9x+3 & ; \ 1/6 \le x < 1/3 \\ 0 & ; \ 1/3 \le x \le 1. \end{cases}$

then $\lim_{n\to\infty} f_n(x) = 0 = f(x) \ ; \ x \in [0,1]$.

V.V.Imp

Pointwise Convergent:- The sequence $\langle f_n(x) \rangle$, $x \in I$ is said to be pointwise convergent on I if every member of family is convergent.

OR, If $\lim_{n\to\infty} f_n(x) = f(x)$, we say sequence $\langle f_n(x) \rangle$ converges to $f(x)$ pointwise if for any $\varepsilon > 0$, $\exists\, m \in \mathbb{N}$ depending on ε and x. both such that

$|f_n(x) - f(x)| < \varepsilon$ whenever $n \ge m$.

OR, $\langle f_n(x) \rangle$ on I and $\forall\, x \in I$ the corresponding sequence of real number convergent then $\langle f_n(x) \rangle$ is convergent pointwise.

Note :- As $\langle f_n(x) \rangle$ is convergent $\forall x \in I$

Hence for each $x \in I$, $\exists\ l_x \in \mathbb{R}$ s.t.

$$\langle f_n(x) \rangle \longrightarrow l_x$$

define $f : I \to \mathbb{R}$

s.t. $f(x) = l_x$

then $\lim_{n\to\infty} f_n(x) = f(x)$.

How to show Pointwise Convergent

Algorithm :-

Step I :- $|f_n(x) - f(x)| \leq \phi(x, n)$

and then write $\phi(x, n) < \varepsilon$.

Step II :- Solve $\phi(x, n) < \varepsilon$ to obtain $n > \psi(x, \varepsilon)$

Step III :- If $\psi(x, \varepsilon)$ is defined for all $\varepsilon > 0$

and $x \in I$ then $m = \lceil \psi(x, \varepsilon) \rceil$

is the desired m.

NET

Problem :- $f_n(x) = x^n$; $x \in [0, 1]$

Show that $f_n(x)$ is pointwise convergent but not uniformly convergent.

Solution :- $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1 & x = 1 \\ 0 & x \in [0, 1) \end{cases}$

Now $|f_n(x) - f(x)| = |x^n - 0| = |x^n| < \varepsilon$

$\Rightarrow n \log x < \log \varepsilon$

$\Rightarrow -n \log x > -\log \varepsilon$

$\Rightarrow n \log (1/x) > \log (1/\varepsilon)$.

$\Rightarrow \quad n > \dfrac{\log(1/\varepsilon)}{\log(1/x)} = \psi(x, \varepsilon)$

Let $m = \lceil \psi(x,\varepsilon) \rceil = \left\lceil \dfrac{\log(1/\varepsilon)}{\log(1/x)} \right\rceil$ → depends on ε and x both

$\therefore$ pt wise cgt

i.e. $n > m$

Hence $|f_n(x) - f(x)| < \varepsilon$ whenever $n > m$.

At $x = 1$

then $\langle f_n(1) \rangle = \langle 1, 1, 1, \ldots 1, \ldots \rangle$

which is constant sequence → also convergent

Hence $\langle f_n(x) \rangle$ is pointwise convergent $\forall x \in [0,1]$.

Hence proved

Remark :- We can not claim the sequence is not uniformly convergent by this method. We claim only pointwise convergence.

Uniform Convergent :- Let $\lim_{n\to\infty} f_n(x) = f(x)$.

We say $\langle f_n(x) \rangle$ converges to $f(x)$ uniformly if for any $\varepsilon > 0$, $\exists\, m \in \mathbb{N}$ depends only ε (independent of x)

s.t. $|f_n(x) - f(x)| < \varepsilon$, $n > m$

How to show Uniform Convergent

Algorithm

Step I :- $|f_n(x) - f(x)| \le \phi(x, n)$ and write $\phi(x,n) < \varepsilon$

Step II :- Solve $\phi(x,n) < \varepsilon$ and obtain $n > \psi(x, \varepsilon)$

Step III :- Try to find $\sup_{x \in I} \psi(x, \varepsilon) \le M(\varepsilon)$

Claim $n > \lceil M(\varepsilon) \rceil = m$.

NET

Problem:- $f_n(x) = x^n$; $x \in [0, a]$, $a < 1$.

To show $\langle f_n(x) \rangle$ is uniformly convergent.

Solution:- $\lim_{n \to \infty} f_n(x) = f(x) = 0$

Now $|f_n(x) - f(x)| = |f_n(x) - 0| = |x^n - 0| = x^n < \varepsilon$

$n \log x < \log \varepsilon$

$\Rightarrow -n \log x > -\log \varepsilon$ $\quad$ $\log x$ is increasing function on $(0, \infty)$

$\Rightarrow n > \frac{\log 1/\varepsilon}{\log 1/x} \Rightarrow n > \left(\frac{\log 1/\varepsilon}{\log 1/a}\right)$

$n > m = \left\lceil \frac{\log 1/\varepsilon}{\log 1/a} \right\rceil$ → depends only on ε. $\Rightarrow$ U.C

Hence $\langle f_n(x) \rangle$ is uniformly convergent in $[0, a]$., $a < 1$

NET

Problem:- $f_n(x) = \frac{x}{1 + nx}$; $x \in (0, \infty)$ or $x \in \mathbb{R}$

To show $\langle f_n(x) \rangle$ is uniformly convergent.

Solution:- $\lim_{n \to \infty} f_n(x) = 0 = f(x)$

$|f_n(x) - f(x)| = \left| \frac{x}{1 + nx} - 0 \right| < \varepsilon$

$\Rightarrow \frac{x}{|nx|} < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow n > m = \lceil \frac{1}{\varepsilon} \rceil$.

Hence $|f_n(x) - f(x)| < \varepsilon$, $n > m$

i.e. $\langle f_n(x) \rangle$ is uniformly convergent in $(0, \infty)$ or $\mathbb{R}$.

U.P.S.C. 08

Problem:- $f_n(x) = n(x^{1/n} - 1)$. Find $f(x)$.

Solution:- $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n(x^{1/n} - 1)$

$$= \lim_{n\to\infty} \frac{x^{1/n} - 1}{1/n}$$

$$= \lim_{t\to 0} \frac{x^t - 1}{t}$$

$$= \lim_{t\to 0} \frac{x^t \log x}{1} = \log x.$$

i.e. $\boxed{f(x) = \log x}$

V. Imp. **Formula**

$\lim_{x\to a} [f(x)]^{g(x)}$; where $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$

then $\lim_{x\to a} [f(x)]^{g(x)} = e^{\lim_{x\to a} [f(x)-1]\, g(x)}$

Problem:- $\lim_{x\to 0} \left(\frac{a_1^x + a_2^x + \cdots + a_n^x}{n}\right)^{1/x} = ?$ (1^∞ form)

Solution:- Here $\lim_{x\to 0} \frac{(a_1^x + a_2^x + \cdots + a_n^x)}{n}$

$$= \frac{1 + 1 + \cdots + 1 \text{ } n \text{ times}}{n} = \frac{n}{n} = 1.$$

and $\lim_{x\to 0} \frac{1}{x} = \infty$

then $\lim_{x\to 0} \frac{(a_1^x + a_2^x + \cdots + a_n^x)}{n}^{1/x}$

$$= e^{\lim_{x\to 0} \left[\frac{a_1^x + a_2^x + \cdots + a_n^x}{n} - 1\right] \cdot \frac{1}{x}}$$

$$= e^{\lim_{x\to 0}\left[\frac{(a_1^x-1)+(a_2^x-1)+\cdots+(a_n^x-1)}{n}\right]\cdot\frac{1}{x}}$$

$$= e^{\frac{1}{n}\lim_{x\to 0}\left[\frac{a_1^x-1}{x}+\frac{a_2^x-1}{x}+\cdots+\frac{a_n^x-1}{x}\right]}$$

$$= e^{\frac{1}{n}(\log a_1+\log a_2+\cdots+\log a_n)}$$

$$= e^{\log(a_1 a_2\cdots a_n)^{1/n}}$$

$$= (a_1 a_2\cdots a_n)^{1/n} \quad \underline{\text{Ans}}$$

Note:- $\boxed{\lim_{x\to 0}\frac{a^x-1}{x} = \log a}$

Problem:- $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x} = ?$

Solution:- Here $\lim_{x\to\frac{\pi}{2}}\sin x = 1$ $\quad\lim_{x\to\frac{\pi}{2}}\tan x = \infty$

Then $\lim_{x\to\frac{\pi}{2}}(\sin x)^{\tan x} = e^{\lim_{x\to\frac{\pi}{2}}[\sin x - 1]\cdot\tan x}$

$$= e^{\lim_{x\to\frac{\pi}{2}}\left[\frac{\sin^2 x-\sin x}{\cos x}\right]}$$

$$= e^{\lim_{x\to\pi/2}\left(\frac{2\sin x\cdot\cos x-\cos x}{-\sin x}\right)}$$

$$= e^0$$

$$= 1 \quad \underline{\text{Ans}}$$

Problem:- (A): $f(x) = x^3$ has no local minima and local maxima

(R): $f'(x) \geq 0$

Solution:- $f'(x) \geq 0$ represents no local maxima or local minima because it does not change its sign.

Then both (A) and (R) are true.

M_n - test (Necessary as well as sufficient)

Let $\lim_{n\to\infty} f_n(x) = f(x)$

Define $M_n = \sup_{x\in I} |f_n(x) - f(x)|$

then sequence $\langle f_n(x)\rangle$ converges to $f(x)$ uniformly iff $\boxed{\lim_{n\to\infty} M_n = 0}$

Problem:- $f_n(x) = \frac{nx}{1+n^2x^2}$; $x \in [0,\infty)$

Test for uniformly convergent.

Solution:- Now $\lim_{n\to\infty} f_n(x) = 0 = f(x)$

Now, again $M_n = \sup_{x\in I} |f_n(x) - f(x)|$

$$= \sup_{x\in[0,\infty)} \left| \frac{nx}{1+n^2x^2} - 0 \right|$$

Let $y = \frac{nx}{1+n^2x^2}$

$$y' = \frac{(1+n^2x^2)n - nx\cdot 2n^2x}{(1+n^2x^2)^2} = \frac{n-n^3x^2}{(1+n^2x^2)^2}$$

[Not U.C.]

$(1-nx)^2 \geq 1+n^2x^2 - 2nx \geq 0$

$\Rightarrow 1+n^2x^2 \geq 2nx$

putting $x = \frac{1}{n}$

$\frac{n}{2n} \geq \frac{nx}{1+n^2x^2}$

$\sup_{x\in[0,\infty)} \frac{nx}{1+n^2x^2} \geq \frac{1}{2} \neq 0$

Put $y' = 0 \Rightarrow x = \pm \frac{1}{n}$

then y has max at $x = \frac{1}{n}$

then $y_{max} = \frac{1}{2}$

$x = \frac{1}{n} \rightarrow$ max^m point

$x = \frac{-1}{n} \rightarrow$ min^m point

$M_n = \frac{1}{2} \Rightarrow \lim_{n\to\infty} M_n = \frac{1}{2} \neq 0$

Hence $f_n(x) = \frac{nx}{1+n^2x^2}$ is not uniformly convergent in $[0,\infty)$.

M.Imp

Problem:- $f_n(x) = nxe^{-nx^2}$; $x \in (0,\infty)$

Test for uniformly convergent. [Not U.C] ✓

Solution:- $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{e^{nx^2}} = \lim_{n\to\infty} \frac{x}{e^{nx^2}\cdot x^2}$

$= 0 = f(x)$

Now, $M_n = \sup_{x\in(0,\infty)} |f_n(x) - f(x)|$

$= \sup_{x\in(0,\infty)} |nxe^{-nx^2} - 0|$

Let $y = nxe^{-nx^2}$

$y' = ne^{-nx^2} + nxe^{-nx^2}(-2nx)$

$\Rightarrow y' = ne^{-nx^2}[1 - 2nx^2]$

$\Rightarrow y' = \frac{n}{e^{nx^2}} \cdot \frac{1}{2n}\left(\frac{1}{\sqrt{2n}} - x\right)\left(\frac{1}{\sqrt{2n}} + x\right)$

Put $y' = 0 \Rightarrow x = \pm \frac{1}{\sqrt{2n}}$

y has max^m at $x = \pm \frac{1}{\sqrt{2n}}$

i.e. $y_{max} = n \cdot \frac{1}{\sqrt{2n}} \cdot e^{-n} \cdot \frac{1}{2n} = \frac{\sqrt{n}}{\sqrt{2}} \cdot \frac{1}{\sqrt{e}}$

i.e. $M_n = \sqrt{\frac{n}{2e}} \Rightarrow \lim_{n\to\infty} M_n = \infty \neq 0.$

Hence $f_n(x) = nxe^{-nx^2}$; $x \in (0,\infty)$ is not uniformly convergent.

M.9mp

Problem:- $f_n(x) = \frac{x^n}{n}$; $x \in (-\infty, \infty)$ Not V.C

Test for uniformly convergent.

Solution:- $\lim_{n\to\infty} f_n(x) = \begin{cases} \infty & ; |x|>1 \\ 0 & ; |x|\leq 1 \end{cases} = f(x)$

Hence $f_n(x)$ is not uniformly convergent for $|x|>1$ because in this case $f(x)$ is ∞.

i.e. $f_n(x) = \frac{x^n}{n}$; $x \in (-\infty, \infty)$ is not uniformly convergent.

Special Case If interval $[-1,1]$ i.e. $|x| \leq 1$.

Hence $\lim_{n\to\infty} f_n(x) = 0 = f(x)$

Now, $M_n = \sup_{x\in[-1,1]} |f_n(x) - f(x)| = \sup_{x\in[-1,1]} \left| \frac{x^n}{n} - 0 \right|$

Let $y = \frac{x^n}{n}$

$y' = x^{n-1}$ | Put $x^{n-1} = 0 \Rightarrow x = 0$

Hence $f_n(x) = \frac{x^n}{n}$; $|x| \leq 1$ is U.C

Note:- Global max. = max. $\{f(a), f(b), f(x_i)\}$

Where $x = x_i$'s are point of local maximum.

Global max. = max. $\{f(-1), f(1), f(0)\}$ = sup $|f_n(x) - f(x)|$

max. $\{-\frac{1}{n}, \frac{1}{n}, 0\}$ = sup $|f_n(x) - f(x)|$

i.e sup $|f_n(x) - f(x)| = \frac{1}{n}$

$M_n = \frac{1}{n}$

Now, $\lim_{n\to\infty} M_n = 0$

Hence $f_n(x)$ is uniformly convergent on $[-1, 1]$.

gmp

Problem:- $f_n(x) = \tan^{-1}(nx)$; $x \in [0, 1]$

show that $f_n(x)$ is pointwise convergent but not uniformly convergent and in $[a, 1]$ $a > 0 \to$ U.C.

Solution:- Here $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} \pi/2 & ; x \in (0, 1] \\ 0 & ; x = 0 \end{cases}$

Now, $|f_n(x) - f(x)| = |\tan^{-1}(nx) - \frac{\pi}{2}| = |\cot^{-1}(nx)| < \varepsilon$

$\Rightarrow nx > \cot\varepsilon$

Since cot is decreasing function

i.e $n > \frac{1}{x}\cot\varepsilon$

$n > m = \left[\frac{\cot\varepsilon}{x}\right]$ $\forall$ $x \in (0, 1]$

i.e $\langle f_n(x) \rangle$ is pointwise convergent in $[0, 1]$.

i.e if $x \in [a, 1]$, $a > 0$

$$n > \frac{\cot\varepsilon}{x}$$

Now, $\frac{\cot\varepsilon}{x} < \left\lceil \frac{\cot\varepsilon}{a} \right\rceil = m$

i.e. $n > m$

Hence $f_n(x) = \tan^{-1}(nx)$ is uniformly convergent in $[a,1]$, $a>0$.

For uniformly convergent

Now by M_n test

$$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$$

$$= \sup_{x\in[0,1]} \left|\tan^{-1}(nx) - \frac{\pi}{2}\right|$$

$$= \sup_{x\in[0,1]} |\cot^{-1}(nx)|$$

Let $y = \cot^{-1}(nx)$

$y' = \frac{-n}{1+n^2x^2}$ · always -ve i.e. strictly decreasing

$\therefore$ $x=0$ gives max^m value.

Then $M_n = \cot^{-1}(0) = \frac{\pi}{2}$

$\lim_{n\to\infty} M_n = \frac{\pi}{2} \neq 0.$

Hence $f_n(x) = \tan^{-1}(nx)$ is not U.C. on $[0,1]$.

- Hence proved.

But in $[a,1]$; $a>0$

Then $M_n = \cot^{-1}(na) \Rightarrow \lim_{n\to\infty} M_n = 0$

$\Rightarrow$ $f_n(x) = \tan^{-1}(nx)$; $x\in[a,1]$, $a>0$ is U.C.

Problem :- $f_n(x) = \begin{cases} n^2x & ; \ 0 \le x < \frac{1}{n} \\ -n^2x + 2n & ; \ \frac{1}{n} \le x < \frac{2}{n} \\ 0 & ; \ \frac{2}{n} \le x \le 1 \end{cases}$ [not U.C]

Test for uniformly convergent on $[0,1]$.

Solution :- $\lim_{n\to\infty} f_n(x) = f(x) = 0 \ ; \ x \in [0,1]$.

i.e. Now $M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$

$$= \max \left\{ \sup_{0\le x\le 1/n} |n^2x - 0|, \sup_{\frac{1}{n}\le x\le \frac{2}{n}} |-n^2x + 2n - 0|, \sup_{\frac{2}{n}\le x\le 1} |0-0| \right\}$$

$$= \max \left\{ n^2 \cdot \frac{1}{n}, -n^2 \cdot \frac{1}{n} + 2n, 0 \right\}$$

$$= \max \{ n, n, 0 \}$$

i.e. $M_n = n$

$\lim_{n\to\infty} M_n = \infty \neq 0$.

Hence $f_n(x)$ is not uniformly convergent.

Problem :- $f_n(x) = \frac{x}{1+nx}$ (U.C)

Test for uniformly convergent on $\mathbb{R}$.

Solution :- $\lim_{n\to\infty} f_n(x) = 0 = f(x)$

Now, $M_n = \sup_{x\in I} |f_n(x) - f(x)| = \sup_{x\in I} \left| \frac{x}{1+nx} - 0 \right|$

Let $y = \frac{x}{1+nx} \Rightarrow y' = \frac{1}{(1+nx)^2} > 0$ always +ve

So y is strictly increasing function

So y is maximum at $x \to \infty$

Then $y_{max} = M_n = \frac{1}{n}$

$$\lim_{n\to\infty} M_n = 0$$

Hence $f_n(x) = \frac{x}{1+nx}$ is U.C. on $\mathbb{R}$.

M.9mp.

Problem:- $f_n(x) = e^{-nx}$; $x \in (0,\infty)$ (Not U.C)

Test for uniformly convergent.

Solution:- $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 1 ; & x = 0 \\ 0 ; & x > 0 \end{cases}$

Now, $M_n = \sup_{x\in(0,\infty)} |f_n(x) - f(x)|$

$= \sup_{x\in(0,\infty)} |e^{-nx} - 0|$

Let $y = e^{-nx}$

$y' = -ne^{-nx} = \frac{-n}{e^{nx}} < 0$

ie y is decreasing function

y is maximum at $x = 0$

then $y_{max} = M_n = 1$

$$\lim_{n\to\infty} M_n = 1 \neq 0$$

Hence $f_n(x) = e^{-nx}$ is not U.C. in $(0,\infty)$.

But $[a,\infty)$; $a > 0$ then $M_n = e^{-na} \Rightarrow \lim_{n\to\infty} M_n = 0 \Rightarrow$ U.C in $[a,\infty)$; $a > 0$

Problem:- $f_n(x) = \begin{cases} |nx-1| & ; 0 \le x < 2/n \\ 1 & ; 2/n \le x \le 1. \end{cases}$

Test for uniformly convergent on $[0,1]$.

Solution:- $f_n(x) = \begin{cases} 1-nx & ; 0 \le x < 1/n \\ nx-1 & ; 1/n \le x < 2/n \\ 1 & ; 2/n \le x \le 1. \end{cases}$

$$\lim_{n\to\infty} f_n(x) = 1.$$

$$\int_0^1 \lim_{n\to\infty} f_n(x) = \int_0^1 1\,dx = [x]_0^1 = 1$$

$$\int_0^1 f_n(x)\,dx = \int_0^{1/n} (1-nx)\,dx + \int_{1/n}^{2/n} (nx-1)\,dx + \int_{2/n}^{1} 1\,dx$$

$$= \left[x - \frac{nx^2}{2}\right]_0^{1/n} + \left[\frac{nx^2}{2} - x\right]_{1/n}^{2/n} + [x]_{2/n}^{1}$$

$$= \frac{1}{n} - \frac{n}{2}\cdot\frac{1}{2n} + n\cdot\frac{4}{2n^2} - 1 - n\cdot\frac{1}{2n^2} + 1 - 1 - \frac{2}{n}$$

$$\Rightarrow \int_0^1 f_n(x)\,dx = \lim_{n\to\infty} \int_0^1 |nx-1|\,dx = 1$$

Since $\int_0^1 \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = 1$

So, we can not say about uniform convergent of $f_n(x)$

So, we now apply M_n-test

$$M_n = \sup |f_n(x) - f(x)|$$

$$= \sup \begin{cases} |1-nx-1| & ; 0 \le x < 1/n \\ |nx-1-1| & ; 1/n \le x < 2/n \\ |1-1| & ; 2/n < x < 1. \end{cases}$$

$$M_n = \sup \begin{cases} 1 & ; \ 0 \le x < 1/n \\ 0 & ; \ 1/n \le x < 2/n \\ 0 & ; \ 2/n \le x \le 1 \end{cases}$$

$\Rightarrow M_n = 1 \neq 0$

$\Rightarrow$ The given sequence $\langle f_n(x) \rangle$ is not U.C.

(All properties are sufficient conditions)

Properties of Uniform Convergence:- If sequence $\langle f_n(x) \rangle$ converges to $f(x)$ uniformly on $[a,b]$ then

1). If $f_n(x)$ is continuous $\forall\, n \in \mathbb{N}$, $x \in [a,b]$ then $f(x)$ is also continuous $\forall\, x \in [a,b]$

In fact.

$$\lim_{x \to a} \left(\lim_{n \to \infty} f_n(x) \right) = \lim_{n \to \infty} \left(\lim_{x \to a} f_n(x) \right) = f(a)$$

i.e. If each $f_n(x)$ is continuous at all $x \in I$ then $f(x)$ is continuous on I.

2). If $f_n(x)$ is differentiable $\forall\, n \in \mathbb{N}$ and $x \in I$, and converges to $f(x)$ uniformly then $\langle f_n'(x) \rangle$ need not to be converge uniformly but whenever if $\langle f_n'(x) \rangle$ converge to some function $g(x)$ then $f'(x) = g(x)$

i.e. If each $f_n(x)$ is differentiable $\forall\, x \in I$ then sequence $\langle f_n'(x) \rangle$ need not converge uniformly.

But if $\langle f_n'(x) \rangle \xrightarrow{\text{uniformly}} g(x)$ then f is differentiable

$$\langle f_n(x) \rangle \longrightarrow \langle f(x) \rangle$$
$$f'(x) \longrightarrow g(x)$$

3). If each $f_n(x)$ is integrable (Riemann) over $[a,b]$ then f is integrable over $[a,b]$.

i.e. If sequence $\langle f_n(x)\rangle$ converge uniformly to $f(x)$ then sign of integration and limit can be interchanged.

i.e. $$\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Problem:- $$f_n(x) = \begin{cases} n^2x & ; \ 0 \le x < 1/n \\ -n^2x + 2n & ; \ 1/n \le x < 2/n \\ 0 & ; \ 2/n \le x \le 1. \end{cases}$$

Test the integral property.

Solution:- Now, $$\int_0^1 f_n(x)\,dx = \int_0^{1/n} n^2x\,dx + \int_{1/n}^{2/n} (-n^2x + 2n)\,dx$$

$$= n^2 \cdot \frac{1}{2n^2} + \left[-n^2 \cdot \frac{4}{2n^2} + 2n \cdot \frac{2}{n} + n^2 \cdot \frac{1}{2n^2} - 2n \cdot \frac{1}{n}\right]$$

$$= \frac{1}{2} - 2 + 4 + \frac{1}{2} - 2$$

$$= 1$$

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 1.$$

Again $$\int_0^1 \lim_{n\to\infty} f_n(x)\,dx = \int_0^1 0\,dx = 0$$

i.e. $$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 \lim_{n\to\infty} f_n(x)\,dx$$

So $f_n(x)$ is not uniformly convergent.

$$\boxed{\begin{array}{c} p \Rightarrow q \\ \sim q \Rightarrow \sim p \end{array}}$$

4). If $\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \alpha$

then $\lim_{x\to a} f(x) = \alpha$.

5). define $g_n(x) = \int f_n(x)\,dx$

then $\langle g_n(x) \rangle \rightleftharpoons \int f_n(x)\,dx$.

Series of Functions

Cauchy Sequence:- Let $\langle f_n(x) \rangle$ be a sequence of functions of $x \in I$, for any $\varepsilon > 0$ $\exists$ p (free from x) $\in \mathbb{N}$

s.t. $|f_{n+p}(x) - f_p(x)| < \varepsilon$

$\forall\, n \in \mathbb{N}$, $x \in I$

Then $\langle f_n(x) \rangle$ is called Cauchy sequence.

Series of functions:-

Definition:- Let $\langle u_n(x) \rangle$ be a sequence of functions then $\sum u_n(x)$ is defined as series of function.

Define $S_n(x) = \sum_{r=1}^{n} u_r(x)$:

Then $\langle S_n(x) \rangle$ is called S.O.P.S. of $\sum u_n(x)$.

Convergence of Series of function

Defn:- 'A series $\sum u_n(x)$ converges pointwise or uniformly according as its S.O.P.S. converges pointwise or uniformly.

i.e. $S_n(x)$ is Cauchy.

Result:- Series of real numbers can be considered as series of constant function and whenever the series and the sequence of real number is convergent it is supposed to be uniformly convergent.

Problem:- Let $u_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$; $[0,\infty)$

Test for $\sum u_n(x)$ is uniformly convergent. (Not U.C)

Solution:- $S_n(x) = \sum_{r=1}^{n} u_r(x)$

$= u_1 + u_2 + \cdots + u_n$

i.e. $S_n(x) = \frac{nx}{1+n^2x^2}$

$\langle S_n(x) \rangle = \left\langle \frac{nx}{1+n^2x^2} \right\rangle$

$(1-nx)^2 \geq 0$
$\Rightarrow 1+n^2x^2 - 2nx \geq 0$
$\Rightarrow 1+n^2x^2 \geq 2nx$
$\Rightarrow S_n(x) \leq \left[\frac{nx}{2nx}\right]$
$= \frac{1}{2} \neq 0$

So $\langle S_n(x) \rangle$ is not uniformly convergent.

Hence $\sum u_n(x)$ is not uniformly convergent.

→ If $\langle f_n(x) \rangle$ and $\langle g_n(x) \rangle$ are sequences of bounded functions on A and converge uniformly on A then $\langle f_n(x) \cdot g_n(x) \rangle$ is U.C. and converge to $f \cdot g$ where f and g are uniform limits of $\langle f_n(x) \rangle$ & $\langle g_n(x) \rangle$.

→ If $\langle f_n(x) \rangle$ and $\langle g_n(x) \rangle$ are U.C. but their product may not be U.C.

Eg: $f_n(x) = x + \frac{1}{n}$; $x \in \mathbb{R}$, $f(x) = x$

$f_n^2(x)$ is not U.C.

Some Important Test

I. **Weierstrass M-test**:- Let $\sum u_n(x)$ be a given series s.t. $|u_n(x)| \leq M_n \ \forall\ x \in I$ then if $\sum M_n$ is convergent then $\sum u_n(x)$ converges uniformly.

Imp.
Problem:- $\sum \frac{\sin nx}{n^\alpha}$, $\alpha > 1$; $x \in \mathbb{R}$.

Test the uniformly convergent on $\mathbb{R}$.

Solution:- Now $|u_n(x)| = \left|\frac{\sin nx}{n^\alpha}\right| \leq \frac{1}{n^\alpha} \ \forall\ x \in I$

Here $M_n = \frac{1}{n^\alpha} \Rightarrow \sum M_n = \sum \frac{1}{n^\alpha}$ is convergent when $\alpha > 1$.

Hence $\sum u_n(x)$ is uniformly convergent on $\mathbb{R}$.

Imp.
Problem:- $\sum \frac{x}{x^p + x^2 n^q}$; $x \in [a, b]$

(i) $p > 1$, $q \geq 0$ (ii) $0 < p \leq 1$, $p + q > 1$.

Test the uniformly convergent.

Imp
Problem:- $u_n(x) = \frac{e^n x^{2^n - 1}}{1 + x^{2n}}$; $x \in \left[\frac{-1}{2}, \frac{1}{2}\right]$

$\sum u_n(x)$ is uniformly convergent or not?

Imp
Problem:- $\sum \frac{(-1)^n x^{2n}}{n^p (1 + x^{2n})}$; $p > 1$.

Test the uniformly convergent.

Uniformly Bounded :- Sequence $\langle f_n(x) \rangle$ is said to be uniformly bounded if $\exists M$ s.t. $|f_n(x)| \leq M \ \forall\, x \in I$ and $n \in \mathbb{N}$.

→ Sequence $\langle f_n(x) \rangle$ is uniformly bounded if set of bounds (lower and upper) of the member of family is bounded.

II. **Dirichlet Test** (It is more stronger than Weierstrass M-test)

Let $\sum u_n(x)\, v_n(x)$ be the series of function such that $\langle v_n(x) \rangle$ is monotonically decreasing with $\lim_{n\to\infty} v_n(x) = 0$ i.e. $\langle v_n(x) \rangle$ is monotonic and converging uniformly to zero and S.O.P.S. of $\sum u_n(x)$ is uniformly bounded (a bound which is independent of x) then $\sum u_n(x).\, v_n(x)$ converges uniformly.

S.O.P.S. $= \langle S_n(x) \rangle$, $S_n = \sum_{r=1}^{n} u_r(x)$

Problem :- $\sum \frac{\sin nx}{n^\alpha}$; $0 < a \leq x \leq b < \pi$, $\alpha > 0$

Solution :- Let $v_n(x) = \frac{1}{n^\alpha}$, $u_n(x) = \sin nx$

① $\langle v_n(x) \rangle \downarrow$ and $\lim_{n\to\infty} v_n(x) = 0$ i.e. $\langle v_n(x) \rangle \to 0$

② Now, $|S_n(x)| = \left| \frac{\sin\left(\frac{nx}{2}\right).\, \sin\frac{(n+1)x}{2}}{\sin\left(\frac{x}{2}\right)} \right|$

$$\leq \frac{1}{\sin\left(\frac{x}{2}\right)} \leq \operatorname{cosec}\frac{a}{2}$$

i.e. $S_n(x)$ is uniformly bounded.

then $\sum \frac{\sin nx}{n^\alpha}$ is uniformly convergent.

Problem:- $\sum (-1)^n \frac{1}{n}$ $\forall$ $x \in I$

Solution:- Let $v_n(x) = \frac{1}{n}$ and $u_n(x) = (-1)^n$

$\langle v_n(x) \rangle \longrightarrow 0$

$S_n(x) = 0$ or -1 as n is even or odd

i.e $S_n(x)$ is uniformly bounded.

then $\sum (-1)^n \frac{1}{n}$ is uniformly convergent.

Remark:- $\sum (-1)^n a_n$; $\langle a_n \rangle \longrightarrow 0$

$a_{n+1} < a_n$

Leibnitz test is particular case of Dirichlet test.

Problem:- $\sum a_n$ is absolutely convergent of real number then show that

$$\sum \frac{a_n x^{2n}}{1+x^{2n}}$$ is uniformly convergent.

Solution:- $\left| \frac{a_n x^{2n}}{1+x^{2n}} \right| \leq |a_n|$

and $|a_n|$ is absolutely convergent

then $\sum \frac{a_n x^{2n}}{1+x^{2n}}$ is U.C. [By Weierstrass M-test].

Interview Ask

$$\sum_{n=1}^{\infty} b^n \cos(a^n \pi x) = f(x)$$ [Uniformly continuous]

(i) $0 < b < 1$ (ii) $a = 2r-1$, $r \in N$ (iii) $ab > 1 + \frac{3\pi}{2}$

then

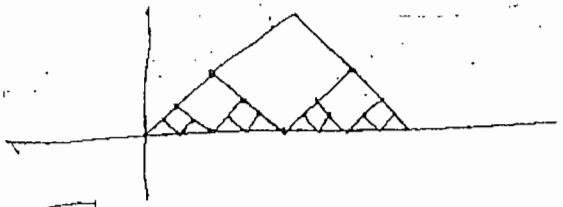
(A) function $f(x)$ is said to be Weierstrass function V.V.Imp

It is continuous on $\mathbb{R}$ i.e. it is continuous everywhere in $\mathbb{R}$ but nowhere differentiable.

Another form of Weierstrass function

$$\sum_{k=1}^{\infty} 2^{-k} \cos(3^k x) = \text{Uniformly continuous}$$

= Throughout continuous on $\mathbb{R}$ but nowhere differentiable on $\mathbb{R}$



Interview

$$f(x) = \sum \frac{\sin n^2 x}{n^2}$$ (It is uniformly continuous)

$f(x)$ is continuous on $\mathbb{R}$ but not known whether it is differentiable or not.

Interview

$$f(x) = \begin{cases} x^2 & ; x \in Q \\ -x^2 & ; x \in Q^c \end{cases}$$

Differentiable only one point (Riemann function)

Some Important Results :-

If $\sum u_n(x)$ converges to $u(x)$ uniformly then

(i) $\lim_{x\to a} \sum u_n(x) = \sum \lim_{x\to a} u_n(x) = \lim_{x\to a} u(x)$.

(ii) If each term is continuous function then sum of functions is continuous.

(iii) $\int_a^b \sum u_n(x)\,dx = \sum \int_a^b u_n(x)\,dx$

or even for indefinite integral.

i.e. sign of summation and integration can be interchanged.

eg: $\int \frac{e^x}{x}\,dx = \int \frac{1}{x}\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)dx$

$= \int \frac{1}{x}\,dx + \int 1\,dx + \int \frac{x}{2!}\,dx + \cdots$

$$\boxed{\int \frac{e^x}{x}\,dx = \log x + \sum_{r=1}^{\infty} \frac{x^r}{r \cdot r!}}$$

(iv) If $\sum u_n(x)$ converges uniformly to $u(x)$ and $u_n(x)$ is differentiable $\forall\, n \in \mathbb{N}$, $\forall\, x \in I$ and $\sum u_n'(x)$ converges uniformly to $g(x)$ then

$$\boxed{u'(x) = g(x)}$$

(v) If each $u_n(x)$ is integrable then $\sum \int u_n(x)\,dx$ converges uniformly to $\int u(x)\,dx$.

III. **Abel's Test**:- If $u_n(x)$ is positive, monotonic decreasing function of n for each fixed value of x in the closed interval $[a, b]$ and $u_n(x)$ is bounded for all values of n and x concerned and if the series $\sum v_n(x)$ is uniformly convergent on $[a, b]$ then so also is the series $\sum u_n(x) v_n(x)$

Problem:- Show that $\sum \frac{a_n x^{2n}}{1+x^{2n}}$; $x \in [0, 1]$ is uniformly convergent if $\sum a_n$ is absolutely cgt.

Solution:- As $\left|\frac{a_n x^{2n}}{1+x^{2n}}\right| < |a_n|$ $\because \frac{x^{2n}}{1+x^{2n}}$ is bdd by 1.

Since $\sum |a_n|$ is uniformly convergent

$\Rightarrow \sum \frac{a_n x^{2n}}{1+x^{2n}}$ is uniformly convergent [By Abel's Test].

Problem:- Show that $\sum (-1)^n \frac{x}{1+nx}$; $x \in [0, 1]$ is uniformly convergent.

Solution:- Since $\sum \frac{x}{1+nx}$ is uniformly cgt on $[0, 1]$.

& $\sum (-1)^n$ is bdd by 1.

So $\sum (-1)^n \frac{x}{1+nx}$ is uniformly convergent [By Abel's test]

UNIT-7

RIEMANN INTEGRAL

Note:- Throughout the discussion, we consider bounded function in the concerned interval.

Partition of an interval:- Let $[a,b] \subset \mathbb{R}$ be closed interval of finite length then the set $P = \{a = x_0, x_1, x_2, \dots, x_n = b\}$ where $a = x_0 < x_1 < x_2 < \dots < x_n = b$ is defined as a partition of $[a,b]$.

Sub interval:- If $P = \{a = x_0, x_1, x_2, \dots x_n = b\}$ is a partition of $[a,b]$ then it partition an $[a,b]$ into n sub intervals

$[x_{r-1}, x_r]$. $\forall$ $r = 1$ to n

and their length of $[x_{r-1}, x_r]$ is Δx_r

i.e. $\Delta x_r = x_r - x_{r-1}$

$=$ length of r^{th} sub-interval.

Norm of a partition:-

The $\max^m \{\Delta x_1, \Delta x_2, \dots \Delta x_n\} = \|P\|$

$= \text{norm}(P)$

$= \mu(P)$

i.e $\|P\| = \mu(P) = \max_{i=1 \text{ to } n} \{\Delta x_i\}$

i.e. length of the largest sub interval of partition P is defined as norm P.

Refinement of the partition :- Let P and P^* be the partitions of closed interval $[a,b]$ s.t. $P \subset P^*$, then P^* is called refinement of P.

Property :- $\boxed{\mu(P^*) \leq \mu(P)}$

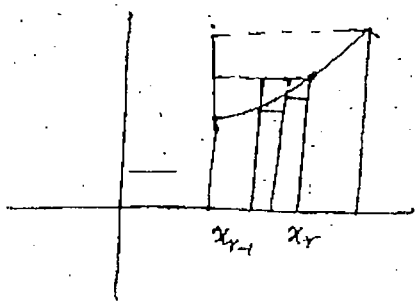
Two important sums regarding a function :- Let $f(x)$ be a bounded function defined on $[a,b]$ and P be a partition of $[a,b]$,

$$P = \{a = x_0, x_1, x_2, \ldots x_n = b\}$$

then Darboux upper sum :-

$$U(f,P) = U(P,f) = \sum_{r=1}^{n} M_r \Delta x_r$$

where $M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$



Area $= M_r \cdot \Delta x_r$

Darboux lower sum

$$L(f,P) = L(P,f) = \sum_{r=1}^{n} m_r \Delta x_r$$

where $m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$

→ Over refinement $U(f,P)$ decreases whereas $L(f,P)$ increases.

Oscillatory sum :- $\sum_{r=1}^{n} (M_r - m_r) \Delta x_r = U(f,P) - L(f,P)$

This is defined as oscillatory sum of f over P.

If $\|P\| \to 0$, oscillatory sum $\to 0$

iff $f(x)$ is R-integrable.

Properties:-

(i) $L(f, P) \leq U(f, P)$ for any partition P.

(ii) If P^* is a refinement of P

(a) $L(f, P^*) \geq L(f, P)$

(b) $U(f, P^*) \leq U(f, P)$

Note: $\sum_{r=1}^{n} \Delta x_r = b-a$

Definitions of Integrability:-

Defⁿ (1):- If $f(x)$ is bounded over $[a,b]$, we say $f(x)$ is integrable over that interval iff the area of the region bounded by $y=f(x)$, $x=a$, $x=b$ and $y=0$ can be obtained finitely.

Defⁿ (2):- $f(x)$ is said to be Riemann integrable over $[a,b]$ iff for any $\varepsilon>0$, $\exists\ \delta>0$ s.t. $|U(f,P) - L(f,P)| < \varepsilon$ for any partition P with $\|P\| < \delta$.

Defⁿ (3):- A bounded function is said to be Riemann integrable over $[a,b]$ iff the oscillatory sum can be made as small as we please by suitable choice of $\|P\|$.

Defⁿ (4): **Riemann definition of Integrability**:- Let P be a partition over $[a,b]$ and $f(x)$ is bounded over $[a,b]$ then f is Riemann integrable

over $[a,b]$ iff $\lim_{\|P\|\to 0} \sum f(t_r)\Delta x_r$ exists finitely.

Where $t_r \in [x_{r-1}, x_r]$ and in all the other cases we symbolize $f \in R[a,b]$

Riemann Upper Integral :-

$$\lim_{n\to\infty} \sum_{r=1}^{n} M_r \Delta x_r = \lim_{\|P\|\to 0} U(f,P) = \overline{\int_a^b} f(x)\,dx.$$

Riemann Lower Integral :-

$$\lim_{n\to\infty} \sum_{r=1}^{n} m_r \Delta x_r = \lim_{\|P\|\to 0} L(f,P) = \underline{\int_a^b} f(x)\,dx,$$

Defⁿ (5) :- $f(x)$ is said to be Riemann integrable over $[a,b]$ iff the lower Riemann integral is equal to upper Riemann integral and then we write

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

Problem :- Show that Dirichlet function is not Riemann integrable over $[0,1]$.

Solution :- $f(x) = \begin{cases} 1 & ; x \in Q \\ -1 & ; x \in Q^c \end{cases}$

$$P = \{0 = x_0, x_1, x_2, \cdots, x_n = 1\}$$

On $[x_{r-1}, x_r]$

$$m_r = -1,$$
$$M_r = 1$$

$$U(f,P) = \sum M_r \Delta x_r = 1 \cdot \sum \Delta x_r = 1(1-0) = 1.$$

$$L(f,P) = \sum m_r \Delta x_r = -1 \sum \Delta x_r = -1(1-0) = -1.$$

Problem :- $f(x) = x \quad ; \quad x \in [0,1]$

$$P = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \cdots \frac{n}{n} = 1\right\}$$

Show that $f(x)$ is Riemann integrable over $[0,1]$.

Solution :- $\text{On } \left[\frac{x_{r-1}}{n}, \frac{x_r}{n}\right]$

$$M_r = \frac{x_r}{n}$$

$$m_r = \frac{x_{r-1}}{n}$$

$$U(f,P) = \sum M_r \Delta x_r = \sum_{r=1}^{n} \frac{r}{n} \cdot \left(\frac{r}{n} - \frac{r-1}{n}\right)$$

$$= \frac{1}{n^2} \sum_{r=1}^{n} r \cdot 1 = \frac{n(n+1)}{2n^2}$$

$$\lim_{\|P\| \to 0} U(f,P) = \lim_{n \to \infty} \sum M_r \Delta x_r = \frac{1}{2}$$

Same as $L(f,P) = \sum m_r \Delta x_r = \sum \frac{r-1}{n} \cdot \frac{1}{n} = \frac{1}{n^2} \cdot \frac{n(n-1)}{2}$

$$\lim_{\|P\| \to 0} L(f,P) = \lim_{n \to \infty} \sum m_r \Delta x_r = \frac{1}{2}$$

$$\int_0^1 f(x)\,dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

Hence $f(x)$ is Riemann integrable over $[0,1]$.

NET

Result:- If f is monotonic on $[a,b]$ then f is Riemann integrable but not conversely. $f(x)=\sin x ; x\in[0,\pi]$

Proof:- We shall prove the result when f is monotonic increasing (the proof for the other case is analogous)

Clearly f is bounded

Let $\varepsilon>0$ be given

Let us choose a number $\eta < \dfrac{\varepsilon}{f(b)-f(a)+1}$

We now, choose a partition $P=\{x_0, x_1, x_2 \cdots x_n\}$ of $[a,b]$ with norm $\mu(P)<\eta$

$M_r = f(x_r)$

$m_r = f(x_{r-1})$; $(r=1,2,\cdots n)$

$$\therefore U(f,P)-L(f,P)=\sum_{r=1}^{n}(M_r-m_r)\Delta x_r$$

$$=\sum_{r=1}^{n}\{f(x_r)-f(x_{r-1})\}\Delta x_r$$

$$<\eta\sum_{r=1}^{n}\{f(x_r)-f(x_{r-1})\}$$

$$<\frac{\varepsilon}{f(b)-f(a)+1}\{f(b)-f(a)\}$$

Hence f is integrable.

Note:- $f(b)-f(a)+1$ has been taken to cover the case when $f(a)=f(b)$.

Eg:- (i) $f(x)=x$ is $\uparrow$ so R-integrable

(ii) $f(x)=[x]$, $x\in[-5,5]$ is $\uparrow$ so R-integrable

(iii) $f(x)=x^2$ is $\uparrow$ so R-integrable $; x\in[5,7]$

Result:- If f is continuous on $[a,b]$ then f is Riemann integrable.

Proof:- $P = \{a = x_0, x_1, x_2, \ldots x_n = b\}$

$[x_{r-1}, x_r]$:

$\sum (M_r - m_r)\Delta x_r$

as f is continuous on $[x_{r-1}, x_r]$

$\exists\ t_r$ and $t_r' \in [x_{r-1}, x_r]$.

s.t. $M_r = f(t_r)$ & $m_r = f(t_r')$

$\sum (M_r - m_r)\Delta x_r = \sum [f(t_r) - f(t_r')]\Delta x_r$

Now, f is uniformly continuous on $[x_{r-1}, x_r]$

then for any $\varepsilon > 0$, $\exists\ \delta_r > 0$

s.t. $|f(y_1) - f(y_2)| < \varepsilon$

$\forall\ y_1, y_2 \in [x_{r-1}, x_r]$; $|y_1 - y_2| < \delta_r$

Choose $\delta = \min\{\delta_1, \delta_2, \ldots \delta_n\}$

$\therefore \sum (M_r - m_r)\Delta x_r \leq \sum [f(t_r) - f(t_r')]\Delta x_r$

$< \sum \frac{\varepsilon}{b-a}\Delta x_r$

$\mu(P) < \delta$

$= \frac{\varepsilon}{b-a}\sum \Delta x_r = \varepsilon$

$\therefore \sum (M_r - m_r)\Delta x_r < \varepsilon$

Hence f is Riemann integrable.

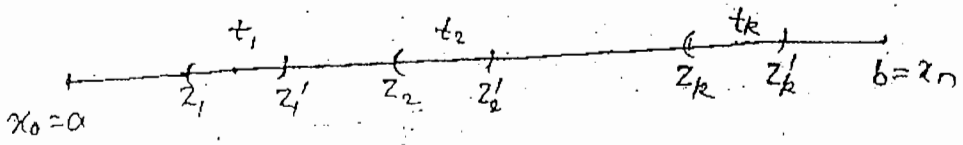
Result:- If f is continuous on $[a,b]-S$, where S is subset of $[a,b]$ then

(A) if S is finite then f is Riemann integral.

(B) if S is countably infinite then f is Riemann integral.

(C) if S has finite number of limit points then f is Riemann integrable.

Proof (A) f is continuous on $[a,b]-S$



$S = \{t_1, t_2, \cdots t_k\}$ $\quad \Delta x_r = x_r - x_{r-1}$.

$\left(t_i - \frac{\varepsilon}{4k}, t_i + \frac{\varepsilon}{4k}\right) = (z_i, z_i')$

$(x_0, z_1)\ (z_1, z_1')\ (z_1', z_2)\ (z_2, z_2')\ \cdots\ (z_k, z_k')$

$$\sum (M_r - m_r)\Delta x_r = \underbrace{\sum (M_r - m_r)\Delta y_r}_{\text{Oscillatory sum of } f \text{ over continuous points}} - \sum (M_r - m_r)\Delta z_r$$

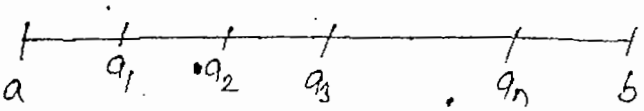
$$< \frac{\varepsilon}{2} + \sum (M + M)\Delta z_r$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e. $\sum (M_r - m_r)\Delta x_r < \varepsilon$.

(B) As S is countably infinite then

$S = \{a_1, a_2, \cdots a_n, a_{n+1}, \cdots\} \subset [a,b]$



Let a_i encloses in (z_i, z_i')

s.t. $\Delta z_i = z_i' - z_i = \frac{\varepsilon}{2^{i+1}}$

$$\sum \Delta z_i = \varepsilon \sum \frac{1}{2^{i+1}} = \frac{\varepsilon}{2}$$

(c) f_ε is discontinuous in S and S has finite number of limit points say k

viz $l_1, l_2, \ldots$

Let $T_i = (l_i - \varepsilon, l_i + \varepsilon)$

$T = \cup T_i$

There is no limit point of S in $[a,b] - T$.

and there are only finite points of S in $[a,b] - T$.

M.Imp

Result:- If f is Riemann integrable on $[a,b]$ and $S \subset [a,b]$ and if $f = g$ $\forall$ $x \in [a,b] - S$ then

(a) If S is finite / countably infinite then g is Riemann integrable on $[a,b]$.

(b) $$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx.$$

Proof:- (a) Let $h(x) = f(x) - g(x)$

$h(x) = 0$, $x \in [a,b] - S$

Hence $h(x)$ is continuous on $[a,b] - S$ and S is finite / countably infinite.

$h \in R[a,b]$

NOW, $f \in R[a,b] \Rightarrow f-h \in R[a,b]$

$\exists\ g \in R[a,b]$.

(5) $L(h,P) = \sum m_r \Delta x_r$

$= \sum m_r \Delta x + \sum m_r' \Delta x'$

$< M \cdot \Delta x'$

$< M \cdot \frac{\varepsilon}{M}$

$< \varepsilon$

$\Rightarrow \lim_{\mu(P)\to 0} L(h,P) = 0$

$\int_a^b h(x)\,dx = 0$

$\Rightarrow \int_a^b (f(x) - g(x))\,dx = 0$

$\Rightarrow \boxed{\int_a^b f(x)\,dx = \int_a^b g(x)\,dx}$

Note:- g has to be bounded, if g is not bounded then g is not Riemann integrable.

→ Riemann integrable होने के लिए function का continuous होना जरूरी नहीं है।

Problem :- Show that the Thoamy function / Riemann function / Porcorn function is Riemann integrable on $[0,1]$ and find Riemann integrable.

Solution :-
$$f(x) = \begin{cases} 1/q & x = p/q \in [0,1]. \\ 0 & x = 0 \text{ or } x \in Q^c \cap [0,1]. \end{cases}$$

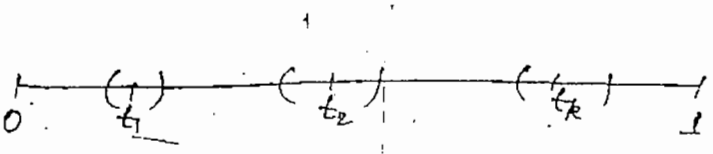
s.t. $f \in R[0,1]$

Let $\varepsilon > 0$, then $\exists\ m \in \mathbb{N}$

s.t. $\frac{1}{n} < \frac{\varepsilon}{2}$ $\forall\ n > m$.

Then there are only finite number of rationals in $[0,1]$ with denominator less than or equal to m.

viz $t_1, t_2, t_3, \dots, t_k$.



$t_1 \in (z_1, z_1') = T_1$

$t_2 \in (z_2, z_2') = T_2$

$t_k \in (z_k, z_k') = T_k$

$T = \cup T_i$

$\Rightarrow [0,1] - T$ has no rational with denominator less than equal to m.

$$\Sigma (M_r - m_r)\Delta x_r = \Sigma (M_r - m_r)\Delta y_r + \Sigma (M_r' - m_r')\Delta z_r$$

$$< \Sigma \frac{\varepsilon}{2}\Delta y_r + \Sigma (1-0)\Delta z_r$$

$$< \frac{\varepsilon}{2}(1-0) + \Sigma \Delta z_r$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

NOTE:- Let $f \in R[a,b]$

and $P = \{a = x_0, x_1, x_2, \ldots x_n = b\}$

$$m \le f(x) \le M$$

and in $[x_{r-1}, x_r]$ if $\sup f(x) = M_r$

and $\inf f(x) = m_r$

then $m \le m_r \le M_r \le M$ for $r = 1$ to n

as $\Delta x_r > 0$

$$\sum m \Delta x_r \le \sum m_r \Delta x_r \le \sum M_r \Delta x_r \le \sum M \Delta x_r \quad ; \text{for } r = 1 \text{ to } n.$$

$$\boxed{m(b-a) \le L(f,P) \le U(f,P) \le M(b-a)}$$

Making $\lim \mu(P) \to 0$

$$\boxed{m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)}$$

$$m \le \frac{1}{b-a} \int_a^b f(x)\,dx \le M$$

If $\frac{1}{b-a} \int_a^b f(x)\,dx = \mu$

$$\boxed{m \le \mu \le M}$$

μ = area per unit length.

$$\boxed{\int_a^b f(x)\,dx = \mu(b-a)} \quad ; \text{Where } m \le \mu \le M$$

If f is continuous on $[a,b]$ then $\exists\, c \in [a,b]$ s.t. $f(c) = \mu$

$$\boxed{\int_a^b f(x)\,dx = f(c)(b-a)}$$

; where μ denotes the area of region bold by $x=a$, $x=b$, x-axis per unit length of interval.

Problem: Stablize $\frac{b-a}{\sqrt{1-a^2}} \leq \sin^{-1}(b) - \sin^{-1}(a) \leq \frac{b-a}{\sqrt{1-b^2}}$

Solution:- Construct $f(x) = \frac{1}{\sqrt{1-x^2}}$, $M = \frac{1}{\sqrt{1-b^2}}$, $m = \frac{1}{\sqrt{1-a^2}}$

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$\frac{b-a}{\sqrt{1-a^2}} \leq \int_a^b \frac{dx}{\sqrt{1-x^2}} \leq \frac{b-a}{\sqrt{1-b^2}}$$

$$\frac{b-a}{\sqrt{1-a^2}} \leq \sin^{-1}(b) - \sin^{-1}(a) \leq \frac{b-a}{\sqrt{1-b^2}}$$

Algebra of Riemann Integrability:-

1). Sum and difference of Riemann integrable function is also Riemann integrable

2). Moduli of Riemann integrable function is also Riemann integrable

3). Product and Ratio of Riemann integrable function is also Riemann integrable, provided ratio is bounded.

Result:- If $f \in R[a,b]$

Define $g(x) = \int_a^x f(t)\,dt$ → may or may not be continuous.

then g is continuous.

Proof:- $g(x+h) = \int_a^{x+h} f(t)\,dt$

Now, $g(x+h) - g(x) = \int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt$

$$= \int_x^a f(t)\,dt + \int_a^{x+h} f(t)\,dt = \int_x^{x+h} f(t)\,dt$$

$$|g(x+h) - g(x)| = \left|\int_x^{x+h} f(t)\,dt\right| \le \int_x^{x+h} |f(t)|\,|dt|$$

$$\le M \int_x^{x+h} |dt|$$

$$\le Mh < \varepsilon$$

$$h = \frac{\varepsilon}{M} = \delta$$

$\therefore$ g is continuous. Hence proved

Primitive:- If $g'(x) = f(x)$ then we say $g(x)$ is primitive of $f(x)$.

Result:- If f is continuous on $[a,b]$ $\xrightarrow{f\in R[a,b]}$ then

$$g(x)=\int_a^x f(t)\,dt$$ is differentiable

and then $g'(x)=f(x)$

i.e. continuous function has primitive.

discontinuous function can have primitive.

Proof:- $$\frac{g(x+h)-g(x)}{h}=\frac{1}{h}\left[\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt\right]$$

$$=\frac{1}{h}\int_x^{x+h} f(t)\,dt$$

Now, $$\frac{g(x+h)-g(x)}{h}+f(x)=\frac{1}{h}\int_x^{x+h} f(t)\,dt-\frac{f(x)}{h}\int_x^{x+h} dt$$

Again now, $$\left|\frac{g(x+h)-g(a)}{h}-f(x)\right|=\left|\frac{1}{h}\int_x^{a+h}(f(t)-f(x))\,dt\right|$$

f is uniformly continuous on $[a,b]$

for any $\varepsilon>0$, $\exists\ \delta>0$

s.t $|f(x_1)-f(x_2)|<\varepsilon$, $|x_1-x_2|<\delta$

$h<\delta$

$$\frac{1}{h}\int_x^{x+h}|f(t)-f(x)|\,dt<\frac{1}{h}\varepsilon\int_x^{x+h} dt=\varepsilon,\quad |h|<\delta$$

i.e $$\left|\frac{g(x+h)-g(x)}{h}-f(x)\right|<\varepsilon$$

$\Rightarrow$ $\boxed{g'(x)=f(x)}$ Hence proved

Fundamental Theorem of Integral Calculus :-

If F is differentiable on (a,b) and continuous on $[a,b]$ and also $F'(x) = f(x)$ and $f \in \mathcal{R}[a,b]$

then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

i.e. If f is integrable on $[a,b]$ & F is primitive of f on $[a,b]$ then $\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$

Proof : F is continuous on $[a,b]$, differentiable on (a,b)

By L.M.V.T. $\exists\ c \in (a,b)$

s.t. $$\frac{F(b) - F(a)}{b-a} = F'(c)$$

Let P be a partition of $[a,b]$

i.e. $P = \{a = x_0, x_1, x_2, \dots x_n = b\}$

and interval $[x_{r-1}, x_r]$

$$\frac{F(x_r) - F(x_{r-1})}{x_r - x_{r-1}} = F'(t_r) \quad ; \quad t_r \in [x_{r-1}, x_r]$$

$$\lim_{\mu(P)\to 0} \sum \left(F(x_r) - F(x_{r-1})\right) = \lim_{\mu(P)\to 0} \sum f(t_r)\,\Delta x_r$$

$$\Rightarrow \boxed{F(b) - F(a) = \int_a^b f(x)\,dx}$$

Hence proved =

Problem:- $f(x) = \begin{cases} 1/q & ; x = p/q \text{ in } [0,1] \\ 0 & ; x = 0 \text{ or } x \in Q^c \text{ in } [0,1]. \end{cases}$

then $f \in R[0,1]$.

Solution:- $\underline{\int} f(x)\,dx = \lim_{\mu(P)\to 0} L(f,P) = \int_a^b f(x)\,dx$

$= \lim_{\mu(P)\to 0} \sum m_r \Delta x_r$ [Since lower value is 0 & $m_r = 0$]

$= 0$

$\int_0^1 f(x)\,dx = 0$

Result:- If f is Riemann integrable on $[a,b]$ then f^2 is also Riemann integrable on $[a,b]$.

Solution:- $U(f^2,P) - L(f^2,P) = \sum (M_r^2 - m_r^2)\Delta x_r$

$= \sum (M_r - m_r)(M_r + m_r)\Delta x_r$

$< 2M \sum (M_r - m_r)\Delta x_r$

$< 2M\varepsilon$

Hence f^2 is also Riemann integrable on $[a,b]$.

Hence proved

Problem:- $f(x) = \lim_{n\to\infty} \lim_{m\to\infty} \left(1 + \cos^{2m} n!\pi x\right)$; $x \in [0,1]$.

Show that $f \notin R[0,1]$.

Solution:- $f(x) = \begin{cases} 2 & ; x \in Q \\ 1 & ; x \in Q^c \end{cases}$

Upper integral = 2, Lower integral = 1

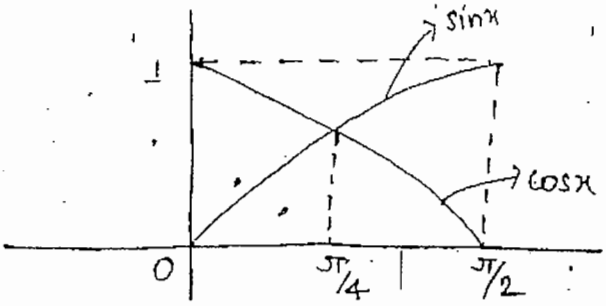
Hence $f \notin R[0,1]$.

V.V. Gmp

Problem :- $f(x) = \begin{cases} \cos x & ; x \in Q \text{ in } [0, \pi/2] \text{ or } x \in Q \cap [0, \pi/2] \\ \sin x & ; x \in Q^c \text{ in } [0, \pi/2] \text{ or } x \in Q^c \cap [0, \pi/2] \end{cases}$

Show that $f \notin R\,[0, \pi/2]$.

Solution :-



$M_r = \cos x_{r-1}$, $m_r = \sin x_{r-1}$ } in $[0, \pi/4]$

and $M_r' = \sin x_r$, $m_r' = \cos x_r$ } in $[\pi/4, \pi/2]$

$$\overline{\int} f(x)\,dx = \lim_{n\to\infty} \left[\sum M_r \Delta x_r + \sum M_r' \Delta x_r \right]$$

$$= \lim_{n\to\infty} \left[\sum_{r=0}^{\pi/4} \cos x_{r-1}\, \Delta x_r + \sum_{r=\pi/4}^{\pi/2} \sin x_r\, \Delta x_r \right]$$

$$= \int_0^{\pi/4} \cos x\,dx + \int_{\pi/4}^{\pi/2} \sin x\,dx$$

$$= \left[\sin x\right]_0^{\pi/4} - \left[\cos x\right]_{\pi/4}^{\pi/2} = \left[\frac{1}{\sqrt{2}} - 0\right] - \left[0 - \frac{1}{\sqrt{2}}\right]$$

$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\underline{\int} f(x)\,dx = \int_0^{\pi/4} \sin x\,dx + \int_{\pi/4}^{\pi/2} \cos x\,dx = -\left[\cos x\right]_0^{\pi/4} + \left[\sin x\right]_{\pi/4}^{\pi/2}$$

$$= -\left[\frac{1}{\sqrt{2}} - 1\right] + \left[1 - \frac{1}{\sqrt{2}}\right] = 2 - \frac{2}{\sqrt{2}} = 2 - \sqrt{2}$$

$\therefore \overline{\int} f(x)\,dx \neq \underline{\int} f(x)\,dx$

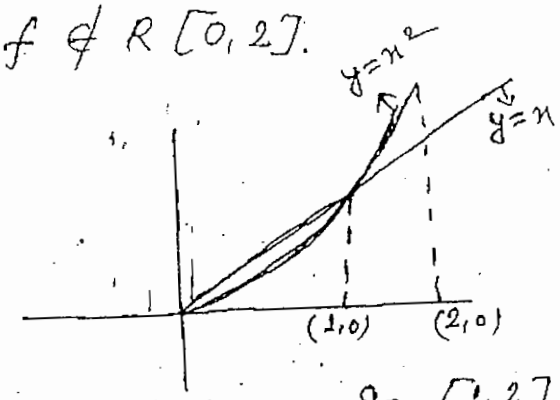
i.e. $f \notin R\,[0, \pi/2]$. Hence proved

V.V. Imp.

Problem :- $f(x) = \begin{cases} x^2 & ; x \in Q \text{ in } [0,2] \text{ or } x \in Q \cap [0,2] \\ x & ; x \in Q^c \text{ in } [0,2] \text{ or } x \in Q^c \cap [0,2] \end{cases}$

Show that $f \notin R[0,2]$.

Solution :-



On $[0,1]$

$M_r' = x_r$

$m_r' = x_{r-1}^2$

On $[1,2]$

$M_r'' = x_r^2$

$m_r'' = x_{r-1}$

Now, $\overline{\int} f(x)dx = \lim_{n\to\infty} \left[\sum M_r \Delta x_r\right]$

$= \lim_{n\to\infty} \left[\sum M_r' \Delta x_r + \sum M_r'' \Delta x_r\right]$

$= \lim_{n\to\infty} \left[\sum x_r \Delta x_r + \sum x_r^2 \Delta x_r\right]$

$= \int_0^1 x\,dx + \int_1^2 x^2 dx = \left[\frac{x^2}{2}\right]_0^1 + \left[\frac{x^3}{3}\right]_1^2$

$= \frac{1}{2} + \frac{7}{3} = \frac{17}{6}$

$\underline{\int} f(x)dx = \int_0^1 x^2 dx + \int_1^2 x\,dx = \left[\frac{x^3}{3}\right]_0^1 + \left[\frac{x^2}{2}\right]_1^2$

$= \frac{1}{3} + \frac{3}{2} = \frac{11}{6}$

i.e. $\overline{\int} f(x)dx \neq \underline{\int} f(x)dx$

Hence $f \notin R[0,2]$ Hence proved

Important Formula

$$\lim_{n\to\infty} \sum_{r=an+b}^{cn+d} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_a^c f(x)\,dx$$

Problem:- $\lim_{n\to\infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n} \right)$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n+r}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{(1+r/n)}$$

$$= \int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1 = \log 2.$$

S. K. RATHORE

Unit-8

Improper Integrals

Proper Integral:- The definite integral $\int_a^b f(x)dx$ is called a proper integral if

(i) the interval of integration $[a,b]$ is finite (or bdd)

(ii) the integrand f is bounded on $[a,b]$.

Improper Integral:- The definite integral $\int_a^b f(x)dx$ is called an improper integral if either the interval is infinite or f is unbdd on $[a,b]$ or neither the interval $[a,b]$ is finite nor f is bdd over it.

OR,

An integral of $f(x)$ over an interval I is said to be improper if either $f(x)$ is unbounded over I or the length of interval is infinite.

OR, $I = \int_a^b f(x)dx$ is said to be improper if

(i) f has point of infinite discontinuities at a or b

(ii) a or b or both is at $+\infty$ or at $-\infty$.

or both (i) & (ii) together.

Types of Improper Integrals:-

Type (I):- In the definite integral $\int_a^b f(x)dx$, if either a or b or both are infinite, so that interval is unbdd. but f is bdd then improper integral of type I or improper integral of first kind.

e.g. :- $\int_1^\infty \frac{dx}{\sqrt{x}}$, $\int_{-\infty}^0 e^{2x}\,dx$, $\int_{-\infty}^\infty \frac{dx}{x^2+2x+2}$

Type II :- In the definite integral $\int_a^b f(x)\,dx$ if a and b are finite but f has one or more point of infinite discontinuity.

i.e f is not bdd on $[a,b]$

(or, f has infinite discontinuity at some $x=c\in I$)

i.e. $\lim_{x\to c} f(x) \longrightarrow \infty$

then $\int_a^b f(x)\,dx$ is called improper integral of type II.

or improper integral of 2nd kind.

e.g :- $\int_0^1 \frac{dx}{x^2}$, $\int_1^2 \frac{dx}{2-x}$, $\int_2^4 \frac{dx}{(x-1)(4-x)}$

→ In $\int_a^b f(x)\,dx$, if the interval is unbounded (i.e. a or b or both are infinite) and f is also unbdd. Then improper integral of 3rd kind.

e.g :- $\int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$

Improper Integral as the Limit of a proper integral

(a) For first kind :-

(1) $\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$; $(t>a)$

(2) $\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$; $(t<b)$

(3) $\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{\infty} f(x)\,dx$ (where 'c' is any real number)

$$= \lim_{t_1 \to -\infty} \int_{t_1}^{c} f(x)\,dx + \lim_{t_2 \to \infty} \int_{c}^{t_2} f(x)\,dx$$

The improper integrals (1), (2) & (3) are said to be convergent if limit on R.H.S exists finitely. and integrals are divergent if limit is $+\infty$ or $-\infty$.

(b) For 2nd kind :-

(1) If $f(x)$ becomes infinite at $x=b$ only

Then $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{b-\varepsilon} f(x)\,dx$

(2) If $f(x)$ becomes infinite at $x=a$ only,

Then $\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_{a+\varepsilon}^{b} f(x)\,dx$

(3). If $f(x)$ is infinite at $x=c$ only, $a<c<b$.

Then $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$$= \lim_{\varepsilon_1 \to 0^+} \int_a^{c-\varepsilon_1} f(x)\,dx + \lim_{\varepsilon_2 \to 0^+} \int_{c+\varepsilon_2}^{b} f(x)\,dx$$

The improper integral $\int_a^b f(x)\,dx$ in (1), (2) and (3) are said to be convergent if limit on R.H.S. exists finitely otherwise divergent

Tests for Convergence of $\int_a^b f(x)\,dx$ at 'a' :

Theorem (1) :- A necessary and sufficient condition for the convergence of the improper integral $\int_a^b f(x)\,dx$ at 'a', where f is positive on $(a, b]$, is that there exists a +ve number M, independent of $\varepsilon > 0$ s.t.

$$\int_{a+\varepsilon}^{b} f(x)\,dx < M \quad \forall\ \varepsilon \text{ in } (0, b-a)$$

Theorem (2) **Comparision Test I :-**

If f and g are two +ve (or non-negative) functions with $f(x) \le g(x)$ $\forall$ x in $(a, b]$ and 'a' is the only point of infinite discontinuity on $[a, b]$ then

(i) $\int_a^b g(x)\,dx$ is cgt $\Rightarrow$ $\int_a^b f(x)\,dx$ is cgt.

(ii) $\int_a^b f(x)\,dx$ is dgt $\Rightarrow$ $\int_a^b g(x)\,dx$ is dgt.

Theorem (3) **Comparision Test II** (Limit form)

If f and g are two positive (or non-negative) functions on $(a, b]$, 'a' being the only point of infinite discontinuity. and $\lim_{x \to a+} \frac{f(x)}{g(x)} = l \neq 0$, where 'l' is non-zero finite number, then

$\int_a^b f(x)\,dx$ & $\int_a^b g(x)\,dx$ cgs or dgs together.

Theorem (4) :- If f and g are two non-negative functions on $(a, b]$, 'a' being the only point of infinite discontinuity, then

(i) $\lim_{x \to a^+} \frac{f(x)}{g(x)} = 0$ and $\int_a^b g(x)\,dx$ converges

$\Rightarrow \int_a^b f(x)\,dx$ converges.

(ii) $\lim_{x \to a^+} \frac{f(x)}{g(x)} = \infty$ and $\int_a^b g(x)\,dx$ diverges

$\Rightarrow \int_a^b f(x)\,dx$ diverges.

Theorem (5) :- **Comparision Integrals**

(1) Improper integral $\int_a^b \frac{dx}{(x-a)^n}$ is cgt iff $n < 1$.

(2) Improper integral $\int_a^b \frac{dx}{(b-x)^n}$ is cgt iff $n < 1$.

Theorem (6) :- (i) If 'a' is the only point of infinite discontinuity on $(a, b]$ and $\lim_{x \to a^+} (x-a)^M f(x)$ exists and is non-zero finite, then

$\int_a^b f(x)\,dx$ converges iff $M < 1$.

(ii) If 'b' is the only point of infinite discontinuity on $[a, b)$ and $\lim_{x \to b^-} (b-x)^M f(x)$ exists and is non-zero finite, then

$\int_a^b f(x)\,dx$ cgs iff $M < 1$.

Problem:- $I = \int_0^1 \frac{1}{x^p}\,dx$.

Find the conditions on p s.t. integral is cgt.

Solution:- If $p \neq 0$, 0 is the point of discontinuity.

Case I:- If $p<0$ then I is proper form.

Case II: If $p>0$ then I is improper.

$$\lim_{\varepsilon\to 0} \int_{0+\varepsilon}^{1} \frac{1}{x^p}\,dx = \lim_{\varepsilon\to 0} \left[\frac{x^{-p+1}}{-p+1}\right]_{0+\varepsilon}^{1} \quad ; p \neq 1$$

$$= \lim_{\varepsilon\to 0} \left[\frac{1}{1-p} - \frac{\varepsilon^{1-p}}{1-p}\right]$$

Hence for the above limit to be existence the power of ε should be positive

i.e. $1-p>0 \Rightarrow$ **$p<1$**

If $p=1$

$$\lim_{\varepsilon\to 0} \int_{0+\varepsilon}^{1} \frac{dx}{x} = \lim_{\varepsilon\to 0} [\log x]_{\varepsilon}^{1} = \lim_{\varepsilon\to 0} [\log 1 - \log \varepsilon]$$

$$= \infty$$

Hence if $p \geq 1$ then integral <u>divergent</u>

and $p<1$ then integral <u>convergent</u>.

Problem:- $I = \int_0^1 \frac{\sin^n x}{x^m}\,dx$

Find the conditions on n and m such that integral is convergent.

Solution:- $x=0$ is the only point of $I.\infty$.

$$f(x) = \frac{\sin^n x}{x^n} \times \frac{1}{x^{m-n}}$$

Construct $g(x) = \frac{1}{x^{m-n}}$

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 1 \neq 0$$

Hence $\int_0^1 \frac{dx}{x^{m-n}}$ and $\int_0^1 \frac{\sin^n x}{x^m}$ converges or diverges together but $\int_0^1 \frac{dx}{x^{m-n}}$ converge when $m-n < 1$

$\Rightarrow m < 1+n$

Hence $\int_0^1 \frac{\sin^n x}{x^m} dx$ is converge when $m < 1+n$

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Problem:- $\int_0^\infty \frac{x^p}{1+x^q \cos^2 x} dx = \lim_{n \to \infty} \sum_{r=1}^{n} \int_{(r-1)k}^{rk} \frac{x^p}{1+x^q \cos^2 x}$ — (1)

where r = natural no.

k = constant

Solution:- In (1) for $p=1, q=4$

(i) Test the convergence of

$$\int_0^\infty \frac{x\,dx}{1+x^4 \cos^2 x}$$

The integrand is positive for positive values of x but the tests obtained for the convergence of +ve integrands so far, are not applicable.

In order to show the integral is convergent we proceed as follows:

Consider $\int_0^{n\pi} \frac{x\,dx}{1+x^4 \cos^2 x}$ and write

$$\int_0^{n\pi} \frac{x\,dx}{1+x^4 \cos^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^4 \cos^2 x}$$

Now $\forall\ x \in [(r-1)\pi, r\pi]$, we have

$$\frac{x}{1+x^4\cos^2 x} \geq \frac{(r-1)\pi}{1+r^4\pi^4\cos^2 x}$$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^4\cos^2 x} \geq \int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi\,dx}{1+r^4\pi^4\cos^2 x}$$

Putting $x = (r-1)\pi + y$, we see that

$$\int_{(r-1)\pi}^{r\pi} \frac{(r-1)\pi\,dx}{1+r^4\pi^4\cos^2 x} = \int_0^{\pi} \frac{(r-1)\pi\,dy}{1+r^4\pi^4\cos^2\{(r-1)\pi+y\}}$$

$$= \int_0^{\pi} \frac{(r-1)\pi\,dy}{1+r^4\pi^4\cos^2 y}$$

$$= 2(r-1)\pi \int_0^{\pi/2} \frac{dy}{1+r^4\pi^4\cos^2 y}$$

$$= 2(r-1)\pi \int_0^{\pi/2} \frac{\sec^2 y\,dy}{1+\tan^2 y + r^4\pi^4}$$

$$= \frac{2(r-1)\pi}{\sqrt{1+r^4\pi^4}} \tan^{-1}\left(\frac{\tan y}{\sqrt{1+r^4\pi^4}}\right)\Bigg]_0^{\pi/2}$$

$$= \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$$

$$\therefore \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{x\,dx}{1+x^4\cos^2 x} \geq \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$$

Hence, $\lim_{n\to\infty} \int_0^{n\pi} \frac{x\,dx}{1+x^4\cos^2 x} \geq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$

But $\sum_{r=1}^{\infty} \frac{(r-1)\pi^2}{\sqrt{1+r^4\pi^4}}$ is a divergent series $\left(\sim \sum_{r=1}^{\infty} \frac{1}{r}\right)$.

$\therefore \int_0^{\infty} \frac{x\,dx}{1+x^4\cos^2 x}$ is divergent. —

(ii) For $p=0, q=4$

Test the convergence of

$$\int_0^{\infty} \frac{dx}{1+x^4\cos^2 x}$$

Consider

$$\int_0^{n\pi} \frac{dx}{1+x^4\cos^2 x} = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{dx}{1+x^4\cos^2 x}$$

Now $\forall\ x \in [(r-1)\pi, r\pi]$

$$\frac{1}{1+x^4\cos^2 x} \leq \frac{1}{1+(r-1)^4\pi^4\cos^2 x}$$

$$\therefore \int_{(r-1)\pi}^{r\pi} \frac{dx}{1+x^4\cos^2 x} \leq \int_0^{\pi} \frac{dy}{1+(r-1)^4\pi^4\cos^2 y}$$

Where $x = (r-1)\pi + y$

$$= 2\int_0^{\pi/2} \frac{\sec^2 y\,dy}{1+\tan^2 y + (r-1)^4\pi^4}$$

$$= \frac{2}{\sqrt{1+(r-1)^4\pi^4}} \tan^{-1} \frac{\tan y}{\sqrt{1+(r-1)^4\pi^4}} \Bigg|_0^{\pi/2}$$

$$= \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$$

$$\lim_{n\to\infty} \int_0^{n\pi} \frac{dx}{1+x^4\cos^2 x} \leq \lim_{n\to\infty} \sum_{r=1}^{n} \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$$

But $\sum_{r=1}^{\infty} \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$ is a convergent series $\left(\sim \sum_{r=1}^{\infty} \frac{1}{r^2}\right)$.

$\therefore \int_0^{\infty} \frac{dx}{1+x^4\cos^2 x}$ is convergent.

$$\rightarrow f_n(x) = \begin{cases} n^{p+1}x & ; \ 0 \leq x \leq 1/n \\ \frac{1}{x^p} & ; \ \frac{1}{n} < x \leq 1 \end{cases}$$

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0 & ; \ x=0 \\ 1/x^p & ; \ x \in (0,1] \end{cases}$$

is convergent if $p<1$ and divergent if $p>1$

0 is the point of discontinuity and not bounded and not R-integrable.

$$\rightarrow \int_0^1 \frac{\sin^n x}{x^m}\, dx$$

$$f(x) = \frac{\sin^n x}{x^m}$$

$$g(x) = \frac{1}{x^{m-n}} \text{ cgt if } m-n<1$$

→ $\int_a^b f(x)dx$ has only point of discontinuity at $x=a$,

∃ μ s.t

$$\lim_{x\to a} (x-a)^{\mu} f(x) = l \neq 0$$

$\Rightarrow \int_a^b f(x)dx$ is cgt if $\mu < 1$

and dgt if $\mu \geq 1$.

→ $f(x) = \int_0^1 \frac{\sin^n x}{x^m} dx$

$$f(x) = \int x^{m-n} \frac{\sin^n x}{x^m}$$

$f(x)$ is cgt if $m-n<1$

dgt if $m-n \geq 1$

V. Imp → $\int_a^\infty \frac{1}{x^p} dx$

cgt if $p>1$ and

dgt if $p \leq 1$

Dirichlet's Test :-

Let $\int_a^\infty f(x)\,dx$ be an Improper integral and $f(x) = \phi(x)\cdot\psi(x)$

If (i) $\lim_{x\to\infty} \phi(x) = 0$ and $\phi(x)$ is monotonic

(ii) $\left| \int_a^t \psi(x)\,dx \right| \le m \quad \forall\ t \in [a,\infty)$

$\Rightarrow \int_a^\infty f(x)\,dx$ is cgt.

e.g :- $\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx$

$$\left| \int_1^t \sin x\,dx \right| = \left| -\cos x \right|_1^t$$

$$= \left| \cos x \right|_1^t$$

$$= |\cos t - \cos 1| \le 2 \quad ; \ t \in (0,\infty)$$

& $\lim_{x\to\infty} \frac{1}{x} = 0$ & $\frac{1}{x}$ is monotonic

$\Rightarrow \int_a^\infty \frac{\sin x}{x}$ is convergent.

e.g :- $\int_2^\infty \frac{\sin x}{\log x}\,dx$

$\lim_{x\to\infty} \frac{1}{\log x} = 0$

$$\left| \int_2^t \sin x\,dx \right| = \left| -\cos x \right|_2^t = |-\cos t + \cos 2| \le 2$$

$\Rightarrow \int_2^\infty \frac{\sin x}{\log x}\,dx$ is convergent.

e.g :- $\int_0^\infty (\sin x^2)\,dx$

Put $x^2 = t$

$\Rightarrow 2x\,dx = dt \Rightarrow dx = \frac{dt}{2\sqrt{t}}$

$$\int_0^\infty \sin t \cdot \frac{dt}{2\sqrt{t}} = \frac{1}{2}\int_0^\infty \frac{\sin t}{\sqrt{t}}\,dt$$

$$= \frac{1}{2}\int_0^1 \frac{\sin t}{\sqrt{t}}\,dt + \int_1^\infty \frac{\sin t}{\sqrt{t}}\,dt$$

$\lim_{t\to\infty} \frac{1}{\sqrt{t}} = 0$

$$\left|\int_1^t \sin t\,dt\right| = \left|-\cos t\right|_1^t = \left|-\cos t + \cos 1\right| \le 2.$$

$\Rightarrow \int_0^\infty (\sin x^2)\,dx$ is convergent.

$\rightarrow I_1 = \int_a^\infty f(x)\,dx$ s.t. $I_2 = \int_a^\infty |f(x)|\,dx$ is convergent

then we say convergence is absolute.

If I_1 is convergent but I_2 is not then convergence is called conditionally convergent.

Note :- Absolute convergence $\Rightarrow$ Convergence

e.g:- $f(x) = \begin{cases} \frac{1}{x^2} & ; x \in Q \\ -\frac{1}{x^2} & ; x \in Q^c \end{cases}$

$$\int_1^\infty |f(x)|\,dx = \int_1^\infty \frac{1}{x^2}\,dx \quad \because 2 > 1$$

$\Rightarrow \int_1^\infty f(x)\,dx$ is absolutely cgt $\Rightarrow \int_1^\infty f(x)\,dx$ is cgt.

e.g:- $\int_1^\infty \left|\frac{\sin x}{x}\right| dx \geq \sum_{r\pi}^{(r+1)\pi} \frac{|\sin x|}{(r+1)\pi}\,dx$

$$= \sum \int_0^{\pi} \frac{\sin x}{(r+1)\pi}\,dx$$

$$= \sum \frac{1}{(r+1)\pi} \cdot k$$

$\Rightarrow$ divergent.

NET

e.g:- $\int_0^\infty e^{-x^2}\,dx$ Put $x^2 = t$

$2x\,dx = dt$

$dx = \frac{dt}{2\sqrt{t}}$

$$= \int_0^\infty e^{-t} \frac{dt}{2\sqrt{t}}$$

$$= \frac{1}{2} \int_0^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$$

Since $e^{x^2} > x^2$

$\Rightarrow \int \frac{1}{e^{x^2}} < \int \frac{1}{x^2}$

$\downarrow$ cgt

$\Rightarrow \int_0^\infty \frac{1}{e^{x^2}}$ is cgt.

Problem:- Find the conditions for m and n so that $\beta(m,n)$ is convergent where $I = \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Solution:- **Case (I)**:- If $m \geq 1$, $n \geq 1$ then I becomes proper integral.

Case (II):- If $m < 1$ and $n < 1$ then I becomes improper integral.

$$I = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx + \int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx$$

$$I = I_1 + I_2$$

Here $I_1 = \int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx = \int_0^{1/2} \frac{1}{x^{1-m}(1-x)^{1-n}}\,dx$

Here $f(x) = \frac{1}{x^{1-m}(1-x)^{1-n}}$

Construct $g(x) = \frac{1}{x^{1-m}}$

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{(1-x)^{1-n}} = 1 \neq 0$$

Hence $\int_0^{1/2} \frac{1}{x^{1-m}}\,dx$ and $\int_0^{1/2} \frac{1}{x^{1-m}(1-x)^{1-n}}\,dx$ converges or diverges together but $\int_0^{1/2} \frac{dx}{x^{1-m}}$ converges when $1-m < 1$ i.e $\boxed{m > 0}$

Hence $\int_0^{1/2} x^{m-1}(1-x)^{n-1}\,dx$ is converges when $\boxed{m > 0}$

and since $I_2 = \int_{1/2}^{1} x^{m-1}(1-x)^{n-1}\,dx$ is convergent

When $\boxed{m>0}$

i.e $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is converges when $\boxed{m>0,\ n>0}$

Problem:- $I = \int_0^2 \frac{dx}{x^2-4x+3}$

Solution:- $I = \underbrace{\int_0^1 \frac{dx}{x^2-4x+3}}_{I_1} + \underbrace{\int_1^2 \frac{dx}{x^2-4x+3}}_{I_2}$

I_2 is Improper integral.

I_1 is Improper integral at $x=1$.

$$\lim_{\varepsilon\to 0} \int_0^{1-\varepsilon} \frac{dx}{x^2-4x+3} = \lim_{\varepsilon\to 0} \int_0^{1-\varepsilon} \frac{dx}{(x-2)^2-1}$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2}\left[\log\left(\frac{x-2-1}{x-2+1}\right)\right]$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2}\left[\log\left(\frac{1-\varepsilon-3}{1-\varepsilon-1}\right) - \log\left(\frac{-3}{-1}\right)\right]$$

$$= \lim_{\varepsilon\to 0} \frac{1}{2}\left[\log\left(\frac{\varepsilon+2}{\varepsilon}\right) - \log 3\right]$$

$$= \frac{1}{2}\left[\log\left(\frac{1+2/0}{1}\right) - \log 3\right]$$

$$= \infty$$

i.e. I is divergent.

M. Imp

Problem:- $I = \int_0^1 e^{-mx} x^n \, dx$

Solution:- m is any one +ve, −ve or 0.

and $n \geq 0$ then I is proper integral

but $n < 0$ the I is improper integral.

Let $f(x) = e^{-mx} \cdot x^n$

$g(x) = x^n = \frac{1}{x^{-n}}$

$\lim_{x \to 0} \frac{f(x)}{g(x)} = e^{-mx} = 1 \neq 0$

$\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{-n}}$ is convergent if $-n < 1$ i.e. $n > -1$

By comparision test

$\int_0^1 e^{-mx} x^n \, dx$ is convergent if $n > -1$

Imp

Problem:- $I = \int_0^1 \left(\log \frac{1}{x}\right)^m dx$

Solution:- Put $\log(1/x) = t$

$1/x = e^t$

$x = e^{-t}$

$dx = -e^{-t} dt$

$I = -\int_0^\infty t^m e^{-t} dt = \int_0^\infty e^{-t} t^m dt$

$= \int_0^1 e^{-t} t^m \, dt + \int_1^\infty e^{-t} t^m \, dt$

I_1 I_2

If $m \geq 0$ then I_1 is proper integral.

If $m < 0$ then I_1 is improper integral.

$$I_1 = \int_0^1 e^{-t} t^m dt$$

$$f(t) = e^{-t} t^m$$

$$g(t) = t^m$$

$$\lim_{t \to 0} \frac{f(t)}{g(t)} = e^{-t} = 1 \neq 0$$

But $\int_0^1 g(t)dt = \int_0^1 \frac{dt}{t^{-m}}$ is cgt if $-m < 1$ i.e. $m > -1$

then I_1 is cgt if $m > -1$

$$I_2 = \int_1^\infty e^{-t} t^m dt$$

$$e^t > t$$

$$e^{-t} < \frac{1}{t}$$

$$\int_1^\infty e^{-t} t^m dt < \int_1^\infty \frac{t^m}{t} dt = \int_1^\infty \frac{1}{t^{1-m}} dt$$

is cgt if $1 - m > 1 \Rightarrow m < 0$

By comparision test

$\int_1^\infty e^{-t} t^m dt$ is cgt if $m < 0$

Hence $\int_0^1 (\log(1/x))^m dx$ is cgt if $0 > m > -1$

$-1 < m < 0$

$m \in (-1, 0)$

Abel's Test :- If $f(x)$ is bounded and monotonic in $[a, \infty)$ and then $\int_a^\infty f(x)g(x)dx$ is convergent.

An Important Type of Improper Integrals :-

$$I = \int_0^\infty \frac{x^p \, dx}{1 + x^q \cos^2 x}$$

Rule :- $\frac{q}{2} - p > 1$ convergent $(<)$

$\frac{q}{2} - p \leq 1$ divergent $(>)$

e.g :- $I = \int_0^\infty \frac{1}{1 + x^3 \cos^2 x} dx$

Here $p = 0, \; q = 3$

$3/2 - 0 = 3/2 > 1$ (convergent)

e.g :- $I = \int_0^\infty \frac{x}{1 + x^3 \cos^2 x} dx$

Here $p = 1, \; q = 3$

$3/2 - 1 = 1/2 < 1$ (divergent)

NET → $\int_2^\infty \frac{\cos x}{\log x} dx$ is not absolutely convergent.

Functions Of Bounded Variation

Variation:- Let P be the partition of $[a,b]$ and f is bounded function defined on $[a,b]$ then variation of f w.r.t. P is denoted by

$$v(P,f) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$$

If P^* be a refinement of P then

$$v(P,f) \leq v(P^*,f)$$

Let $\mathcal{P}[a,b] = \{$set of partition of $[a,b]\}$

So $\sup_{P \in \mathcal{P}[a,b]} v(P,f) =$ the total variation

$$= v[f,[a,b]]$$

Function of bounded Variation: A function is defined on $[a,b]$ is said to be of bounded variation if its total variation is finite or bounded.

i.e. $\boxed{\sup_{P} v(f,P) < \infty}$

i.e $v(f,P)$ exists finitely then it is function of bounded variation.

Variation Function :- $v(x) = \overset{b}{v}(f, [a,x])$

= total variation of f over $[a,x]$.

M. M. 9mp

Properties :-

(i) $v(a) = 0$

(ii) $v(b)$ = total variation of f over $[a,b]$

(iii) $x_1 < x_2 \Rightarrow v(x_1) \leq v(x_2)$

i.e $v(x)$ is non-decreasing function.

(iv) $v(x)$ is continuous iff $f(x)$ is continuous.

V. 9mp

Results :-

(i) Every monotonic function is of bounded variation if bounded. Converse is not true.

(ii) It is necessary for a function to be of bounded variation if it is bounded not sufficient.

Proof of (i) :- $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$

$f(x_i) \leq f(x_{i+1})$

$$v(f, P) = \sum_{i=0}^{n} |f(x_{i+1}) - f(x_i)|$$

$$= \sum_{i=0}^{n} f(x_{i+1}) - f(x_i)$$

$$= f(b) - f(a)$$

(iii) If f is differentiable and derivative is bounded then function is of bounded variation but not conversely.

Proof:- $|f(x_r) - f(x_{r-1})| = |f'(t_r)\,\Delta x_r|$

$= |f'(t_r)|\,\Delta x_r$

$\leq M\,\Delta x_r$

$\Sigma |f(x_r) - f(x_{r-1})| \leq M(b-a)$ $\quad [\because \Sigma \Delta x_r = (b-a)]$

Note:- Neither continuity nor uniform continuity along with boundedness confirm a function is of bounded variation.

e.g:- (a) f is defined on $[0,1]$

$$f(x) = \begin{cases} x\sin\frac{1}{x} & ;\ x \neq 0 \\ 0 & ;\ x = 0 \end{cases}$$

f is not bounded variation. It is uniform continuous.

(b) f is defined on $[0,1]$

$$f(x) = \begin{cases} x\sin\frac{\pi}{x} & ;\ x \neq 0 \\ 0 & ;\ x = 0 \end{cases}$$

f is not bounded variation.

V.Imp

Remark:- Uniform continuous $\not\Rightarrow$ Bounded variation

Riemann Integrable $\not\Rightarrow$ Bounded variation

Q.8 :- $$f(x) = \begin{cases} x \cos \frac{\pi}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

$f(x)$ is continuous on $[0,1]$

$\Rightarrow$ $f(x)$ is uniformly continuous.

For bounded variation, we have

Let $P = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n}\}$ be the partition

$$v(f, P) = \sum_{r=1}^{n} |f(x_r) - f(x_{r-1})|$$

$$= \sum_{r=1}^{n} \left|\frac{1}{n}\right| \quad \text{which is dgt} \quad [\because p=1]$$

Thus we can not find the sum.

$\Rightarrow$ f is not of bounded variation

(iv) If f is bounded variation, then it is Riemann integrable. But converse is not true.

(v) Differentiation does not guarantee the bounded variation. It is the boundedness of the derivative which guarantee.

(vi). Let $g(x)$ be Riemann integrable function over $[a,b]$ and $f(x) = \int_a^x g(t)\,dt$, then this f is of bounded variation.

$$\boxed{v(f, [a,x]) = \int_a^x |f'(t)|\,dt}$$

(vii) If $f(x)$ is of bounded variation on $[a,b]$ then it can be expressed as difference of two monotonic non-decreasing function and conversely. Infact this is only class of function of bounded variation.

[This is necessary and sufficient condition].

e.g.: $f(x) = \{x\} \quad ; x \in [0,5]$

$$f(x) = x - [x]$$

↓ ↓

non-decreasing functions

$\Rightarrow f(x)$ is difference of two non-decreasing functions

Hence $f(x)$ is of bounded variation.

Proof: Let f be of bounded variation

$$g(x) = V(f, [a,x])$$

$g(x)$ is increasing.

Construct $h(x) = g(x) - f(x)$

Let $x_1 < x_2$

$$h(x_2) - h(x_1) = g(x_2) - f(x_2) - g(x_1) + f(x_1)$$

$$= \{g(x_2) - g(x_1)\} - \{f(x_2) - f(x_1)\}$$

$$= V(f, [x_1, x_2]) - \{f(x_2) - f(x_1)\}$$

But we know that

$$V(f, [a,b]) \geq f(b) - f(a)$$

$\therefore h(x_2) - h(x_1) \geq 0$

Hence h is non-decreasing function

and $f = g - h$

But g and h are non-decreasing functions

$\therefore$ f is also a non-decreasing function

Hence a function is of B.V. iff it can be expressed as difference of two non-decreasing functions.

(viii) $\overset{b}{v}(f, [a,b]) = v(f, [a,c]) + v(f, [c,b])$; $c \in [a,b]$

Problem:- $f(x) = \sin x$; $x \in [0, 2\pi]$

find the total variation.

Solution:- $v(f, [a,x]) = \int_a^x |f'(t)|\,dt$

$$v(f, [0, 2\pi]) = \int_0^{2\pi} |\cos t|\,dt$$

$$= \int_0^{\pi/2} \cos t\,dt + \int_{\pi/2}^{3\pi/2} -\cos t\,dt + \int_{3\pi/2}^{2\pi} \cos t\,dt$$

$$= [\sin t]_0^{\pi/2} + [-\sin t]_{\pi/2}^{3\pi/2} + [\sin t]_{3\pi/2}^{2\pi}$$

$$= [1-0] + [-1+1] + [0+1]$$

$$= 1+2+1$$

$$= 4$$

i.e. the total variation in $[0, 2\pi] = 4$ Ans

Problem :- $f(x) = \cos x$; $x \in [-\pi/2, \pi/2]$.

. Find the total variation.

Solution : $V(f, [a, x]) = \int_a^x |f'(t)|\,dt$

$$V(f, [-\pi/2, \pi/2]) = \int_{-\pi/2}^{\pi/2} |-\sin t|\,dt$$

$$= 2\int_0^{\pi/2} \sin t\,dt$$

$$= 2\,[-\cos t]_0^{\pi/2}$$

$$= 2\,[-\cos \pi/2 + \cos 0].$$

$$= 2\,(-0+1).$$

$$= 2$$

i.e. the total variation in $[-\pi/2, \pi/2] = 2$ Ans

$\rightarrow$ $$f(x) = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases} \qquad x \in [-1, 1]$$

is of bounded variation.

Problem:- Show that $f(x) = \begin{cases} x \sin\left(\frac{\pi}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ in $[0,1]$

is not of bounded variation.

Solution:- $P = \left\{ \overset{x_0}{0}, \overset{x_1}{\frac{2}{2n+1}}, \cdots\cdots \overset{x_{n-3}}{\frac{2}{7}}, \overset{x_{n-2}}{\frac{2}{5}}, \overset{x_{n-1}}{\frac{2}{3}}, \overset{x_n}{1} \right\}$

$$v(f,[a,b]) = \lim_{n\to\infty} \left[|f(x_1) - f(x_0)| + |f(x_2) - f(x_1)| + \cdots + |f(x_n) - f(x_{n-1})| \right]$$

$$= \lim_{n\to\infty} \left[\text{---} + \left| \frac{2}{3}\sin\frac{2\pi}{3} - \frac{2}{5}\sin\frac{2\pi}{5} \right| + \left| \sin\pi - \frac{2}{3}\sin\left(\frac{2\pi}{3}\right) \right| \right]$$

$$= \lim_{n\to\infty} \left[4\left(\frac{1}{3} + \frac{1}{5} + \text{---} + \frac{1}{2n+1} \right) \right]$$

$$= \lim_{n\to\infty} 4 \sum_{r=1}^{n} \frac{1}{(2n+1)}$$

$$= 4 \sum_{n=1}^{\infty} \frac{1}{2n+1} \longrightarrow \text{divergent}$$

not uniform continuous.

So it is not bounded variation.

Remark:- Total boundedvariation of constant function is zero.

Note:- (i) $f(x) = \begin{cases} x^\alpha \sin\left(\frac{\pi}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $x \in [0,1]$

$f(x)$ is of bounded variation iff $\alpha > 1$

$$P = \left[0, \frac{2}{(2n+1)\pi}, \cdots\cdots \frac{2}{7\pi}, \frac{2}{5\pi}, \frac{2}{3\pi}, \frac{1}{\pi} \right]$$

$$v(f,[a,b]) = 4 \sum_{n=1}^{\infty} \frac{1}{(2n+1)^\alpha} \sim \frac{1}{n^\alpha} \quad \alpha > 1 \text{ cgt}$$

NOTE (ii) $f(x) = \begin{cases} x^\alpha \left(\sin \frac{1}{x}\right)^\beta & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $x \in [0, 1]$

$f(x)$ is of bounded variation iff $\frac{\alpha}{\beta} > 1$ i.e $\alpha > \beta$

$$P = \left\{0, \left(\frac{2}{(2n+1)\pi}\right)^{1/\beta}, \cdots, \left(\frac{2}{7\pi}\right)^{1/\beta}, \left(\frac{2}{5\pi}\right)^{1/\beta}, 1\right\}.$$

NOTE:- Above result is also true for cosx.

→ $f(x) = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Here $f(x)$ is of bounded variation because $\alpha > \beta$

→ Set the conditions for α & β of above functions

Show that

(i) $\lim x \to 0$ exists

(ii) f and g are continuous at $x = 0$

(iii) f' exists

(iv) f' exists and f' is continuous or f' exists and of bounded variation.

(v) f'' exists.

(vi) f is of bounded variation.

Algebra of functions of bounded variation

If f and g are functions of bounded variation then

(i) $|f|$ and $|g|$ are of bounded variations

(ii) $f+g$ and $f-g$ are of bounded variations

(iii) $f \cdot g$ is of bounded variations

V.9mp (iv) f/g need not to be of bounded variations.

$f = \begin{cases} x^2 \sin 1/x & ; x \neq 0 \\ 0 & \end{cases}$ $g = x$, $x \in [-1,1]$

$f/g = x \sin 1/x$ is not b.v on $[-1,1]$

V.9mp → Dirichlet's function and Riemann function are not bounded variations

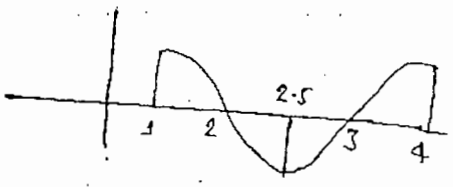
V.9mp → B.V ⇒ Riemann integrable but converse is not true.

i.e Riemann integrable ⇏ B.V.

Eg:- $f(x) = \begin{cases} x \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $x \in [0,1]$.

→ $f(x) = x^2 - 5x + 6$ on $[1,4]$

$f'(x) = 2x - 5$



$$V(f,[1,4]) = \int_1^4 |2x-5|\,dx$$

$$= \int_1^{5/2} (5-2x)\,dx + \int_{5/2}^4 (2x-5)\,dx$$

$$= (5x - x^2)_1^{5/2} + (x^2 - 5x)_{5/2}^4$$

$$= 25/2$$

$$= |f(2.5) - f(1)| + |f(4) - f(2.5)|$$

$$\boxed{V(x) = \int_a^x |f'(t)|\,dt}$$ → V. 9mp formula.

Positive Variation & Negative Variation Functions :-

Let f be B.V. on $[a,b]$.

Define two functions

$$f(x) - f(a) = p(x) - q(x)$$

$$v(x) = p(x) + q(x)$$

These $p(x)$ and $q(x)$ are called positive variation function and negative variation function.

Properties :-

(i) $p(a) = q(a) = 0$

(ii) Functions $p(x)$ and $q(x)$ are non-decreasing functions monotonically.

e.g :- $f(x) = \sin x \quad ; x \in [0, \pi/2]$

$$p(x) = \frac{f(x) + v(x) - f(a)}{2}$$

Here $f(a) = 0$

$$p(x) = \frac{f(x) + v(x)}{2}$$

$$\Rightarrow v(x) = \int_0^x \cos x \, dx$$

$$= \sin x$$

$$\therefore p(x) = \frac{\sin x + \sin x}{2} = \sin x$$

$$\Rightarrow \boxed{p(x) = \sin x} \quad \& \quad \boxed{q(x) = 0}$$

$\rightarrow$ $f(x) = \sqrt{x} \quad ; x \in [0,1]$

$f'(x) = 1/2\sqrt{x}$

If derivative is unbold then function may or may not be bold variation.

V.9mp

→ If $f(x)$ is of bounded variation has to be bounded but converse is not true.

Eg: $f(x) = \begin{cases} x \cdot \cos\frac{\pi}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$ $x \in [0,1]$

This function is not of bounded variation but $f(x)$ is bounded

→ (i) $f(x) = x^2 - 4x^3$

(ii) $f(x) = x - \sin x$

(iii) $f(x) = \{x\}$

All are B.V.S.

Unit-9

Functions of Several Variables.

$f: \mathbb{R}^2 \to \mathbb{R}$

e.g.:- (i) $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$

(ii) $f(x,y) = \sqrt{xy}$

(iii) $f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) & ; y \neq 0 \\ 0. & ; y = 0 \end{cases}$

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

$f(x,y)$ is continuous at $(x,y) = (a,b)$.

Partial derivative:-

$$f_x(a,b) = \frac{\partial f}{\partial x}_{at\,(a,b)} = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}$$

Geometrically when we differentiate partially w.r.t. 'x' keeping y as constant we basically differentiate on the curve obtained by the simultaneous eqn $y = b$ and $z = f(x,y) = x+y$.

Differentiability of $f(x,y)$:- $f(x,y)$ is said to be differentiable at (a,b) if $\exists$ constant α and β depending on the function f and the point (a,b) both s.t.

$$f(a+h, b+k) - f(a,b) - h\alpha - k\beta = h\,\phi_1(h,k) + k\,\phi_2(h,k)$$

$$\text{or} \quad = \sqrt{h^2+k^2}\,\phi_3(h,k)$$

then $\lim_{(h,k)\to(0,0)} \phi(h,k) = 0$.

Result :- The constant α and β mention in the defination of differentiable are the partial derivative of f w.r.t. x and y respectively if $f(x,y)$ is differentiable.

$$\alpha = f_x(a,b)$$

$$\beta = f_y(a,b)$$

Proof :- Let $f(x,y)$ is diff at $(x,y) = (a,b)$

$$f(a+h, b+k) - f(a,b) - h\alpha - k\beta = \sqrt{h^2+k^2}\,\phi(h,k)$$

$(h,k) \to (0,0)$ on x-axis then $\underline{k=0}$:

$$f(a+h, b) - f(a,b) - h\alpha = |h|\,\phi(h,k)$$

$$\frac{f(a+h,b) - f(a,b)}{h} - \alpha = \frac{|h|\,\phi(h,k)}{h}$$

$$\lim_{h\to 0}\left[\frac{f(a+h,b)-f(a,b)}{h} - \alpha\right] = \lim_{h\to 0} \frac{|h|\,\phi(h,k)}{h}$$

$$\lim_{h\to 0}\left[\frac{f(a+h,b)-f(a,b)}{h} - \alpha\right] = 0$$

i.e $$\lim_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h} = \alpha$$

i.e. $f_x(a,b) = \alpha$

Same as $f_y(a,b) = \beta$

Problem :- ① $f(x,y) = \begin{cases} \sqrt{xy} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Check differentiability at $(0,0)$

Solution :- $\alpha = f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$

$\beta = 0$

Now, $f(a+h, b+k) - f(a,b) - h\alpha - k\beta = h\phi_1(h,k) + k\phi_2(h,k)$

Then

$f(h,k) - f(0,0) - 0 - 0 = \sqrt{hk} = h \cdot \sqrt{\frac{k}{h}} + k \cdot 0$

$= h \cdot \phi_1(h,k) + k\phi_2(h,k)$

Here $\phi_1(h,k) = \sqrt{\frac{k}{h}}$, $\phi_2(h,k) = 0$.

put $h = r\cos\theta$
$k = r\sin\theta$
then do
solve

$\lim_{(h,k) \to (0,0)} \phi_1(h,k) = \lim_{(h,k) \to (0,0)} \sqrt{k/h}$

Put $k = mh$

$= \lim_{h \to 0} \sqrt{\frac{mh}{h}} = \sqrt{m}$ which is depend on m.

i.e. limit does not exist

Hence $f(x,y)$ is not differentiable at $(0,0)$.

Problem :- ② $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Check differentiability at $(0,0)$.

Solution :- $\alpha = f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$

$\beta = 0$

Now, $f(h,k) - f(0,0) - h\alpha - k\beta = \frac{hk}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}\left(\frac{hk}{h^2+k^2}\right)$

Here $\phi(h,k) = \frac{hk}{h^2+k^2}$

put $h = r\cos\theta$, $k = r\sin\theta$ then also solve

$\lim_{(h,k) \to (0,0)} \phi(h,k) = \lim_{(h,k) \to (0,0)} hk/h^2+k^2$

Put $k = mh$

$= \lim_{h \to 0} \frac{mh^2}{h^2 + m^2h^2}$

$= \lim_{h \to 0} \frac{m}{1+m^2}$

$= \frac{m}{1+m^2}$

which is depend on m.

Hence limit does not exist.

i.e. $f(x,y)$ is <u>not</u> differentiable at <u>$(0,0)$</u>

Problem:- ③ $f(x,y) = \begin{cases} x\sin\left(\frac{1}{y}\right) + y\sin\left(\frac{1}{x}\right) & ; xy \neq 0, x\neq 0, y\neq 0 \\ 0 & ; xy = 0 \end{cases}$ (other case)

Check differentiability at $(0,0)$.

Solution:- $\alpha = f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$

$\beta = 0$

Now, $f(h,k) - f(0,0) - h\alpha - k\beta = h\cdot\sin\left(\frac{1}{k}\right) + k\sin\left(\frac{1}{h}\right)$

Here $\phi_1(h,k) = \sin\left(\frac{1}{k}\right)$

$\phi_2(h,k) = \sin\left(\frac{1}{h}\right)$

Now, $\lim_{(h,k)\to(0,0)} \phi_1(h,k)$ and $\lim_{(h,k)\to(0,0)} \phi_2(h,k)$ limit does not exist.

Hence $f(x,y)$ is not differentiable at $(0,0)$.

Problem: ④ $f(x,y) = \begin{cases} \frac{x^2y^2}{\sqrt{x^2+y^2}} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Check differentiability at $(0,0)$.

Solution:- $\alpha = f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = 0$

$\beta = 0$

Now, $f(h,k) - f(0,0) - h\alpha - k\beta = \frac{h^2k^2}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}\left(\frac{h^2k^2}{h^2+k^2}\right)$

Here $\phi(h,k) = \frac{h^2k^2}{h^2+k^2}$

Now, $\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} h^2k^2/h^2+k^2$

Put $k = mh$

$$= \lim_{h\to 0} \frac{m^2h^4}{h^2+m^2h^2} = \lim_{h\to 0} \frac{m^2h^2}{1+m^2} = 0$$

Hence $f(x,y)$ is differentiable at $(0,0)$.

Homogeneous function:- $f(x,y)$ is said to be homogeneous function of degree n

if $f(tx, ty) = t^n f(x,y)$ or $x^n f(y/x) = f(x,y)$.

Total differential:- $u = u(x,y)$

Total differential of u is defined as

$$\boxed{du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy}$$

If d is taken as operator

$$\boxed{d^n u = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n u}$$

and $$\boxed{d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + \frac{\partial^2 u}{\partial y^2}(dy)^2 + 2\frac{\partial^2 u}{\partial x \partial y}dx\,dy}$$

We have taken u_x, u_y are continuous and successively differentiable.

Now, all the application in one variable of $\frac{dy}{dx}$ has correspondence with du if function of two variables. Similarly n^{th} order correspondence with n^{th} order total derivative d^n.

Stationary Point:- In one dimension

$$\frac{dy}{dx} = 0$$

and in two dimension, $du = 0$

$$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = 0$$

$\Rightarrow \frac{\partial u}{\partial x} = 0$ and $\frac{\partial u}{\partial y} = 0$

i.e. the root of simultaneous equation $u_x = 0$ and $u_y = 0$ gives the stationary point of u.

$d^2u > 0 \longrightarrow$ Point of local minima

$d^2u < 0 \longrightarrow$ Point of local maxima

at the point $u_x = 0$ and $u_y = 0$ simultaneously.

Problem:- $u(x,y) = x^3 + y^3 - 3axy$

Find local minima and local maxima point.

Solution:- Now, $u_x = 3x^2 - 3ay$

$u_y = 3y^2 - 3ax$

Put $u_x = 0$ and $u_y = 0$

$3x^2 = 3ay$ and $3y^2 = 3ax$

$\Rightarrow x^4 = a^2y^2$

$\Rightarrow x^4 = a^3x \Rightarrow x = 0, a$

Then $y = 0, a, -a$

i.e stationary point $(0,0)$, (a,a), $(a,-a)$

$r = u_{xx} = 6x$

$t = u_{yy} = 6y$

$s = u_{xy} = -3a$

Now, $d^2u = u_{xx}\,dx^2 + u_{yy}\,dy^2 + 2u_{xy}\,dx\,dy$

$= u_{xx}X^2 + u_{yy}Y^2 + 2u_{xy}XY$

$= \begin{bmatrix} u_{xx} & u_{xy} \\ u_{xy} & u_{yy} \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = AX$

Here $A = \begin{bmatrix} r & s \\ s & t \end{bmatrix}$

Note

✓ $rt - s^2 > 0$, $r > 0$ ⟶ minima

✓ $rt - s^2 > 0$, $r < 0$ ⟶ maxima

✓ $rt - s^2 < 0$ ⟶ neither maximum nor minimum

✓ $rt - s^2 = 0$ ⟶ doubtful case.

✓ $d^2u > 0$, if A is +ve definite

i.e. $r > 0$ and $rt - s^2 > 0$ then gives local minima point.

✓ $d^2u < 0$, if A is −ve definite

i.e. $r < 0$ and $rt - s^2 > 0$ then gives local minima point.

Now at (a, a) $r = t = 6a$, $s = -3a$

$r > 0$ $rt - s^2 = 36a^2 - 9a^2 = 27a^2 > 0$

i.e. (a, a) is local minima point

Now at $(0, 0)$ $r = t = 0$, $s = -3a$

$rt - s^2 = -9a^2 < 0$

i.e. $(0, 0)$ is local maxima point.

Euler's Theorem:- If u is homogeneous function in x_i of degree n then $\boxed{\sum_{i=1}^{n} x_i \frac{\partial u}{\partial x_i} = nu}$ where $n \in Q$

or, $\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu}$ for two variables.

Problem:- $u(x, y) = x^2 - 3y^2 + 4xy$

Find $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}$

Solution:- Here u is homogeneous function of degree 2.

Then by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u$$
$$= 2(x^2 - 3y^2 + 4xy) \text{ Ans}$$

Remark:- $u = f(\phi(x,y))$

where $\phi(x,y)$ is homogeneous function of degree n.

$\phi(x,y) = f^{-1}(u)$.

Let $f^{-1}(u) = g(u)$

$\phi(x,y) = g(u) \Rightarrow$ By Euler's eqn

$$x\frac{\partial g}{\partial x} + y\frac{\partial g}{\partial y} = n \cdot g$$

Then $$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{g(u)}{g'(u)}}$$ ← $xg'(u)\cdot\frac{\partial u}{\partial x} + yg'(u)\frac{\partial u}{\partial y} = ng$.

Problem:- $u = \tan^{-1}\left(\frac{x^{1/2} - y^{1/2}}{x^{1/5} + y^{1/5}}\right)$

Find $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$

Solution:- $\tan u = \frac{x^{1/2} - y^{1/2}}{x^{1/5} + y^{1/5}} = \phi(x,y)$ is homogeneous function.

$\uparrow$ $g(u)$

Here $n = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$

Then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\cdot\frac{g(u)}{g'(u)} = \frac{3}{10}\cdot\frac{\tan u}{\sec^2 u}$

$= \frac{3}{10}\sin u \cos u$. Ans

Problem:- $u(x,y) = \log(x^2+y^2)$

find $x\frac{\partial y}{\partial x} + y\frac{\partial y}{\partial y}$

Solution:- $e^y = x^2+y^2 = \phi(x,y)$ is homogeneous function of degree 2.

$\downarrow$

$g(u)$

then $x\frac{\partial y}{\partial x} + y\frac{\partial y}{\partial y} = n\cdot\frac{g(u)}{g'(u)} = 2\cdot\frac{e^y}{e^y}$

$= 2$

i.e. $\boxed{x\frac{\partial y}{\partial x} + y\frac{\partial y}{\partial y} = 2}$ Ans

Real valued functions of n variables

Defⁿ ①:- Let $S \subseteq \mathbb{R}^n$ and $f: S \to \mathbb{R}$

$f(x_1, x_2, \ldots x_n) = y \in \mathbb{R}$, $\alpha = (\alpha_1, \alpha_2, \ldots \alpha_n)$

and α is a limit point of S need not to be a member of S, we say f has limit $l \in \mathbb{R}$ at $X = \alpha$ if for any $\varepsilon > 0$, $\exists\ \delta > 0$ s.t.

$|f(X) - l| < \varepsilon$

whenever $0 < |X - \alpha| < \delta$

i.e. for given $\varepsilon > 0$, we can always find $\delta > 0$

whenever $d(X, \alpha) < \delta$

$\Rightarrow d(f(X), l) < \varepsilon$

and then we write

$\lim_{X \to \alpha} f(X) = l$

or, $\boxed{\lim_{(x_1, x_2, \ldots x_n) \to (\alpha_1, \alpha_2, \ldots \alpha_n)} f(x_1, x_2, \ldots x_n) = l}$ and this limit is called simultaneous limit.

Def"(2):

GPEL :- General Principal of Existence of Limit

We say $\lim_{x \to a} f(x)$ exists if for $\varepsilon > 0$, $\exists\ \delta > 0$

s.t. $|f(x_1) - f(x_2)| < \varepsilon$ $\forall\ x_1, x_2 \in 0 < |x - a| < \delta$

Def" (3) :- Let f be defined on S and a is a limit point of S. Further let $l \in \mathbb{R}$ and $S \subset \mathbb{R}^n$

we say $\lim_{x \to a} f(x) = l$ if whenever $x \to a$ in any manner $f(x) \to l$.

Note :- How to claim limit does not exist.

(i) We say $\lim_{(x,y) \to (a,b)} f(x,y)$ does not exist if there exists two paths $y = \phi(x)$ and $y = \psi(x)$ on which $(x,y) \to (a,b)$ and $f(x,y)$ tends to different values.

e.g. $f(x,y) = \begin{cases} \frac{x^2 - y^2}{x^2 + y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

$f(x,y) = \lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ limit does not exist.

→ $\lim_{(x,y) \to (0,0)} f(x,y)$

$y = mx$ (Choose path)

$\lim_{x \to 0} \phi(x) = f(x, mx)$

In particular we generally choose linear path $y = mx + c$ and for $(0,0)$ we choose $y = mx$.

And if along this path limit comes out to be function of 'm' then we say limit (a,b) does not exist.

(ii) If this limit is free from 'm' then we should not claim that limit does not exist even if limit exist.

(iii) If $\lim_{(x,y)\to(0,0)} f(x,y) = l$ (free from m) along $y = mx$ then we say limit exists along with radius vector.

Algorithm :- How to show limit exists

$$\lim_{(x,y)\to(a,b)} f(x,y) = l$$

Step I :- Simplify $|f(x,y)-l|$ to get the inequality

$$|f(x,y)-l| \le \alpha|x-a| + \beta|y-b| \quad \text{---(1)}$$

OR

$$|f(x,y)-l| \le \alpha|x-a||y-b| \quad \text{---(2)}$$

Step II :- Put $\delta = \frac{\varepsilon}{\alpha+\beta}$ for (1)

$\delta = \sqrt{\frac{\varepsilon}{\alpha}}$ for (2)

Problem :- $f(x,y) = 3x+4y$

Find $\lim_{(x,y)\to(1,2)} f(x,y) = 3x+4y$

Solution :- $3x+4y = 3\times1 + 4\times2 = 11$.

$$|3x+4y-11| \le 3|x-1| + 4|x-2|$$

$$\delta = \frac{\varepsilon}{\alpha+\beta} = \frac{\varepsilon}{3+4}$$

i.e. $\boxed{\delta = \frac{\varepsilon}{7}}$

[S. K. RATHORE]

Problem :- $\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

$y^2 \le x^2+y^2$

$|y| \le \sqrt{x^2+y^2}$

$\frac{|y|}{\sqrt{x^2+y^2}} \le 1$

Solution :- $\left| \frac{xy}{\sqrt{x^2+y^2}} - 0 \right| = \frac{|x|\cdot|y|}{\sqrt{x^2+y^2}}$

$\le |x| + 0\cdot|y|$

$= \therefore |x-0| + 0\cdot|y-0|$

$\boxed{\delta = \varepsilon}$

$\rightarrow$ $f(x,y) = \begin{cases} x\sin\frac{1}{y} + y\sin\frac{1}{x} & ; x\cdot y \neq 0 \quad x\neq 0,\ y\neq 0 \\ 0 & ; x\cdot y = 0 \end{cases}$

Show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Solution :- $\left| x\sin\frac{1}{y} + y\sin\frac{1}{x} - 0 \right| \le |x| + |y|$

$\boxed{\delta = \varepsilon/2}$

Problem :- $f(x,y) = \begin{cases} 1 & ; x\in Q,\ y\in Q^c \\ 2 & ; x\in Q^c,\ y\in Q \\ 3 & ; x\in Q^c,\ y\in Q^c \\ 4 & ; x\in Q,\ y\in Q \end{cases}$

Show that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Solution :- $\langle x_n, y_n\rangle \longrightarrow (a,b)$

if $\langle x_n\rangle \to a$

$\langle y_n\rangle \to b$

$\langle t_n, z_n\rangle \longrightarrow (a,b)$

$x_n \in Q$, $y_n \in Q^c$

$t_n \in Q^c$, $z_n \in Q$

$f(x_n, y_n) \longrightarrow 1$ when $x_n \in Q$, $y_n \in Q^c$

$f(x_n, y_n) \longrightarrow 2$ when $x_n \in Q^c$, $y_n \in Q$

$\therefore$ limit does not exist.

→ $\lim_{(x,y)\to(0,0)} f(x,y) = \sin 1/x$

$\lim_{x\to 0} f(x,0) = \sin 1/x$

limit does not exist.

Repeated Limit :- The limits (i) & (ii) are called repeated limit which is given below.

$\lim_{(x,y)\to(a,b)} f(x,y)$

$z = f(x,y)$

$\lim_{x\to a} f(x,y) = \phi(y)$

$$\lim_{y\to b}\lim_{x\to a} f(x,y) = \lim_{y\to b}\phi(y) = l_1 \quad \text{(i)}$$

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lim_{x\to a}\psi(x) = l_2 \quad \text{(ii)}$$

Results:-

(i) If repeated limit exists does not imply simultaneous limit exist and conversely.

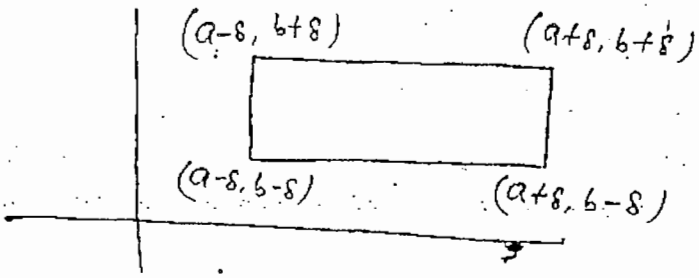
(ii) If simultaneous limit exists along with repeated limits then all these three are equal.

Simultaneous Limit

Defⁿ:- We say the simultaneous limit

$\lim_{(x,y)\to(a,b)} f(x,y) = l$ exists

if for any $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x,y) - l| < \varepsilon$ whenever $|x-a| < \delta$, $|y-b| < \delta$



Algorithm:-

When l is the limit of $f(x)$

$$|f(x,y) - l| = \alpha|x-a| + \beta|y-b|$$
$$\leq \alpha|x-a||y-b|$$
$$\leq \alpha \cdot \delta^2 < \varepsilon$$

$$\Rightarrow \boxed{\delta = \sqrt{\varepsilon/\alpha}}$$

e.g:- $\lim_{(x,y)\to(0,0)} \frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} = 0$

$$\left|\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2} - 0\right| = \left|\frac{\sqrt{1+x^2y^2}-1}{x^2+y^2}\right|$$

$$= \left|\frac{1+\frac{x^2y^2}{2}+\cdots-1}{x^2+y^2}\right|$$

$$= \left|\frac{x^2y^2}{2(x^2+y^2)}\right|$$

$$\leq \left|\frac{x^2}{2}\right| \cdot \left|\frac{y^2}{x^2+y^2}\right|$$

$$< \frac{|x-0|^2 \cdot 1}{2}$$

$$= \frac{\delta^2}{2} = \varepsilon.$$

Continuity

Defn:- We say f is continuous at $x=a \in S$ where S is domain of defination of f

$$\boxed{\lim_{x\to a} f(x) = f(a)}$$

Note:- If $a \in S$ but not a limit point of S then f is assumed to be continuous there at.

→ $$f(x) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & x, y \neq (0,0) \\ x^2 \sin\frac{1}{x} & y = 0,\ x \neq 0 \\ y^2 \sin\frac{1}{y} & x = 0,\ y \neq 0 \\ 0 & (x,y) = (0,0) \end{cases}$$

$$f_x = \lim_{h \to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{h^2 \sin 1/h}{h} = 0 = \alpha$$

$$f_y = \lim_{k \to 0} \frac{f(0,0+k) - f(0,0)}{k} = \lim_{k \to 0} \frac{k^2 \sin 1/k}{k} = 0 = \beta$$

$$f(0+h, 0+k) - f(0,0) = h^2 \sin\frac{1}{h} + k^2 \sin\frac{1}{k} - 0$$

$$= h^2\left(\sin\frac{1}{h}\right) + k^2\left(\sin\frac{1}{k}\right) \to 0$$

as $(h,k) \to (0,0)$

$\Rightarrow$ $f(x,y)$ is differentiable at $(0,0)$.

→ $f(x,y) = \sqrt{xy}$ is continuous at $(0,0)$ but not differentiable at $(0,0)$.

Extremum points of surface :-

$u = u(x,y)$ be the given function then point of extrema are obtained by solving $du = 0$

i.e. solving simultaneous equation $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$.

Let the solutions are (α_i, β_i)

Obtain $\frac{\partial^2 f}{\partial x^2} = A$, $\frac{\partial^2 f}{\partial x \partial y} = H$ & $\frac{\partial^2 f}{\partial y^2} = B$

$$d^2u = \frac{\partial^2 u}{\partial x^2}(dx)^2 + 2\frac{\partial^2 u}{\partial x \partial y} dx\,dy + \frac{\partial^2 u}{\partial y^2}(dy)^2$$

$$\Rightarrow d^2u = AX^2 + 2HXY + BY^2 = [XY]\begin{bmatrix} A & H \\ H & B \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

$$= A\left[X^2 + \frac{2H}{A}XY + \frac{B}{A}Y^2\right]$$

$$= A\left[X^2 + 2X\cdot\frac{HY}{A} + \frac{H^2Y^2}{A^2} - \frac{H^2Y^2}{A^2} + \frac{B}{A}Y^2\right]$$

$$\Rightarrow d^2u = A\left[\left(X + \frac{HY}{A}\right)^2 + \frac{Y^2}{A^2}(BA - H^2)\right]$$

$AB - H^2 > 0$ then

$A > 0 \longrightarrow$ minima

$A < 0 \longrightarrow$ maxima

$A = 0 \longrightarrow$ Neither maxima nor minima

(i) If $AB - H^2 > 0$ i.e $\begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0$ — $f=4$

if $A > 0$, $f(a,b)$ is a minimum value

if $A < 0$, $f(a,b)$ is a maximum value

(ii) If $AB - H^2 < 0$ i.e $\begin{vmatrix} A & H \\ H & B \end{vmatrix} < 0$

then $f(a,b)$ is not an extreme value.

(iii) If $AB-H^2=0$; Further investigation.

(iv) If $A=0$, $B\neq 0$; No extreme value

(v) If $A=0$, $B=0$, Further Investigation.

→ For a function $f(x,y,z)$ of three independent variables, sufficient conditions for (a,b,c) to be an extreme point are that

(i) $df(a,b,c) = f_x\,dx + f_y\,dy + f_z\,dz$ so that $f_x=f_y=f_z=0$ and

(ii) $d^2f(a,b,c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}\,dx\,dy + 2f_{yz}\,dy\,dz + 2f_{zx}\,dz\,dx$

$= AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY$

So, $A = f_{xx}$, $B = f_{yy}$, $C = f_{zz}$

$F = f_{yz}$, $G = f_{zx}$, $H = f_{xy}$

$$AX^2 + BY^2 + CZ^2 + 2FYZ + 2GZX + 2HXY = d^2u$$

$$[X\ Y\ Z]\begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

A Hummy Girl Having Boy Friend Going For Coffee.

A H G H B F G F C

$$\begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}$$

$A>0$, $\begin{vmatrix} A & H \\ H & B \end{vmatrix}>0$, $|B|>0$ → minima

$A<0$ $\begin{vmatrix} A & H \\ H & B \end{vmatrix}>0$, $|B|<0$ → maxima

→ If all the eigen values are +ve then minima and all are -ve then maxima.

→ All the eigen values are greater than or equal to zero then either minima or not a point of extrema.

→ All the eigen values are less than or equal to zero then then either maxima or not a point of extrema.

→ If there is one pair of eigen values is of opposite sign then neither of them is maxima nor minima.

V.Imp

✍ Discontinuous + Monotonic function ⇒ Do not have Primitive.

✍ $f: \mathbb{R} \to \mathbb{R}$

$f(x) = (x^2+1)^{2003}$

(i) one-one onto

(ii) One-one

(iii) Onto.

(iv) neither one-one nor onto

Sol: Since for the +ve or -ve values we get only +ve images. So neither one-one nor onto

$\because f(a) = f(-a)$, $a \in \mathbb{R}$ (-ve real numbers have no preimages.)

✍ कोई ऐसा Sequence जो bdd हो और उसका Subsequence oscillate finitely.

Eg: Sequence: $-3, -2, 1, 2, 1, 2, \ldots$

Subsequence: $1, 2, 1, 2, 1, 2 \ldots$ (Oscillates finitely)

V.Imp

→ The function $f(x) = \frac{1}{1+|x|}$ is continuous and bdd and attains its supremum for $x=0$ but does not attain its infimum.

V.Imp

→ The function $f(x) = \frac{-1}{1+|x|}$ is continuous and bdd and attains its infimum for $x=0$ but does not attain its supremum.

V.Imp

→ The function $f(x) = x \ \forall\ x \in (0,1)$ is continuous and bdd but neither attains the infimum nor the supremum.

→ If f is monotonically in $[a,b]$ then $f'(a) > 0 \Rightarrow f(x) > f(a)$

$f'(a) < 0 \Rightarrow f(x) < f(a)$

$f'(b) > 0 \Rightarrow f(x) < f(b)$

$f'(b) < 0 \Rightarrow f(x) > f(b)$

→
$$f(x) = \begin{cases} x(x-1)(x-2)(x-3) & ; x \in \mathbb{Q} \\ 0 & ; x \notin \mathbb{Q} \end{cases}$$

4 points पर continuous है $(0,1,2,3)$

→
$$f(x) = \begin{cases} x(x-1)(x-2)(x-3)(x-4)(x-5) & ; x \in \mathbb{Q} \\ 0 & ; x \notin \mathbb{Q} \end{cases}$$

6 points पर continuous है $(0,1,2,3,4,5)$

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